Matching models with capacity restriction: Variation of the quota[1]

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Abstract

In this paper I consider a special assignment model which involves two complementary types of agents (type I workers and type II workers) and an institution, which has preferences over the possible matchings. It can hire a set of pairs of complementary workers and has a quota which is the maximum number of candidates allowed to be hired.

I study the effect caused when the institution can increase the quota, and I design both a procedure to find maximal pairs and an algorithm that generates all the stable assignments with a new quota starting from the stable assignments with the previous quota.

Resumen

En este artículo se considera un modelo de asignación especial que implica dos tipos complementarios de agentes (trabajadores del tipo I y trabajadores del tipo II) y una institución, que tiene preferencias sobre los posibles emparejamientos. Se puede contratar a un conjunto de pares de los trabajadores complementarios y tiene una cuota, que es el número máximo de candidatos permitidos a ser contratados.

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Se estudia el efecto causado cuando la institución puede aumentar la cuota y se diseña tanto un procedimiento para encontrar los pares máximos, como un algoritmo que genera todas las asignaciones estables con una nueva cuota a partir de las asignaciones estables con la cuota anterior.

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1 Introduction

Bilateral matching models are used to study problems of markets whose distinctive feature is that the agents involved are members of disjoint sets with different characteristics (e.g., employers and workers, hospitals and physicians. etc.). The nature of the problem studied here consists of assigning each member of a set to one agent of another set at most. Within these models the term *matching* refers to the bilateral nature of the relation between the agents of these two sets.

The fundamental question of this assignment problem consists of the matching of each worker from one side with a worker from the other side. Roth (1984, 1986, 1990 and 1991), Mongell and Roth (1991), Roth and Xing (1994), and Romero-Medina (1997) are examples of research work which investigates particular matching problems like entry-level professional labor markets and student admissions at colleges.

The agents have preferences on the potential partners. Stability has been considered the main property to be satisfied by any sensible matching. A matching is called stable if all the agents are matched to an acceptable partner and there is no matched pair of workers that would prefer another partner to their current one.

The matching model with capacity restriction consists of a model by means of which each worker from one side of the market is assigned one worker on the other side, such that the pairs of workers hired by the institution are q at most, that is to say, the institution will have to choose q pairs of workers at most according to their order of preference.

It should be noted that, even though this model involves two sets of workers and an institution, it is not equivalent to the trilateral allocation model introduced in 1986 by Alkan [1], in which there is no stability.

The property of stability in this model depends on the preferences expressed by the participants and the preference of the institution; for this reason, the property of q-stability is defined. Unfortunately, the set of q-stable matchings may be empty. However, under the restriction of the institution's responsive preferences, the existence of the set of q-stable matchings is guaranteed and it is possible to obtain its characterization. These results have been obtained with Femenia, Neme and Oviedo as joint authors.

We consider a matching model with capacity constraint in which the organization has hired q pairs of workers at most and, for some reason (eg, technological, budgetary, building-related, etc.), its hire quota varies and, as a consequence, a new model is obtained.

In this paper we show the relationship between subsets of stable matchings of these two models. We define the concept of maximal pair for an assignment and show a method to find all maximal pairs for an assignment established beforehand. In addition, we give an algorithm by which, starting from a model with a fixed quota, we can find all other stable assignments in the new model with capacity constraints, in which the given quota increases.

The paper is organized as follows. In Section 2 we present the notation and the definitions of the model. We introduce responsive preferences on assignments. We show that the set of q-stables is not empty and give its characterization. In Section 3 we consider some relationships between subsets of stables of two models with capacity constraints in which the quota is increased. We define a maximal reduced model and, given a model with a fixed quota, we show how to obtain the stables of a model with another quota starting from the set of stables in the original model.

2 Preliminaries

2.1 The model

This model consists of two finite and disjoint sets of agents, i.e. the set of n type I workers and the set of m type II workers , denoted by $D = \{d_1, ..., d_n\}$ and $E = \{e_1, ..., e_m\}$, respectively.

Each worker $d \in D$ has a strict preference relation[2] P_d over the set of agents $E \cup \{\phi\}$, and each worker $e \in E$ has a strict preference relation P_e over the set of agents $D \cup \{\phi\}$.

Notice that only workers' strict preferences are being considered in his work. In some cases, similar results may be obtained if indifference is allowed.

Preference profiles are $(n+m)$-tuples of preference relations represented by $P = (P_{d_1}, ..., P_{d_n}; P_{e_1}, ..., P_{e_m}) = (P_D, P_E)$.

Given a preference profile P, the standard matching market is denoted by $M = (D, E, P)$.

Given a preference relation $P_d(P_e)$**,** the workers in the set preferred to the empty set by $d\,(e)$ are called *acceptables*. Similarly, given a preference relation P_e**,** the workers in the subsets of workers preferred to the empty set by e are called *acceptables*. The assignment problem consists of matching type I workers with type II workers keeping the bilateral nature of their relationship as well as the possibility for both types of workers to remain unemployed. Formally:

Definition 1 *A* ***matching*** μ *is a mapping from set* $D \cup E$ *into set* $D \cup E \cup \{\phi\}$, *such that for all* $d \in D$ *and* $e \in E$:

1. Either $\mu(d) \in E$ or $\mu(d) = \phi$.

2. Either $\mu(e) \in D$ or $\mu(e) = \phi$.

3. $\mu(d) = e$ if and only if $\mu(e) = d$.

[2] A preferenceisabinary, reflexive, antisymmetrical, transitiveand completerelation.

Let M be the set of all possible matchings μ.

Given a matching market $M = (D, E, P)$, a matching μ is blocked by a single agent $f \in D \cup E$ if $\phi P_f \mu(f)$. We say that a matching is *individually rational* if it is not blocked by any single agent. A matching μ is *blocked by a pair* of workers (d, e) if $dP_e\mu(e)$ and $eP_d\mu(d)$.

Definition 2 *A matching μ is* ***stable*** *if it is not blocked by any individual agent or by any pair of workers.*

Given a matching model $M = (D, E, P)$, $S(M)$ denotes the set of stable matchings. Notice that Gale and Shapley (1962) [2] proved that the set of stable matchings is non-empty, i.e., $S(M) \neq \phi$.

2.2 The matching model with quota restriction

It consists of two disjoint sets of agents, the set of n type I workers and the set of m type II workers, denoted by $D = \{d_1, ..., d_n\}$, $E = \{e_1, ..., e_m\}$, respectively, and an institution denoted by U. Institution U has a binary relation R_U over the set of all possible matchings M, the empty matching included. Let P_U and I_U denote the strict and indifferent preference relations induced by R_U, respectively. The pair of workers will work for institution U,which has a maximum number of positions - quota $q \leq \min\{n, m\}$- to be filled; then, only the matchings whose cardinality is smaller or equal to q may be acceptable. The institution may choose some matchings of $\mathcal{M}$ according to its preference P_U and their quota restriction q. We denote $M_q = \ \mu \in M : \#\mu \leq q\}$.

This new matching marker is denoted by $M_U^q = (M, R_U, q)$.

A matching μ is *acceptable* for institution U according to their preferences if $\mu \in M_q$ and $\mu R_U \mu^\phi$, in which μ^ϕ is the matching such that $\mu^\phi(x) = \phi$, for every $x \in D \cup E$.

Given M and a quota $q \leq \min\{n, m\}$, the institution can only accept assignments of M which are most preferred to the empty matching according to its preference P_U, and its cardinal is not larger than the allowed number of positions $\#\mu \leq q$. A matching is acceptable if the partner assigned is preferred to the empty set. Formally,

Definition 3 *Given a model* M_U^q*, an assignment* μ *is* $q-$*individually rational if* $\#\mu \leq q$*,* $\mu P_U \phi$ *and for every* $f \in D \cup E$ *such that* $\mu(f) P_f \phi$ *is verified.*

Given an assignment $\mu \in M$ and a pair of workers $(d, e) \in D \times E$ we define $\mu_{(d,e)}$ as follows:

$$\mu_{(d,e)}(f) = \begin{cases} \mu(f) & \text{if } f \notin d, e, \mu(d), \mu(e) \\ d & \text{if } f = e \\ \phi & \text{otherwise} \end{cases}$$

Notice that, if $\mu(d) = e$, then $\mu_{(d,e)} = \mu$.

Remark 4 *The matching* $\mu_{(d,e)}$ *may not be individually rational. Let us consider a matching* μ *such that* $\#\mu = q$*and let*(d, e) *such that, if* $\mu(d) = \phi = \mu(e)$*, then* $\#\mu_{(d,e)} > q$*and* $\mu_{(d,e)}$ *is not* $q-$*individually rational.*

Usually, in standard models, (d, e) is a blocking pair if these agents are not assigned to each other and if they each prefer one another to their current partners. Note that in our model, we may have a blocking pair (d, e) such that the matching that the blocking pair is satisfying is not acceptable for institution U. Then, we will consider two types of blocking pairs for μ. One type is that which occurs when the assignment μ is blocked by a couple of agents in the institution, already assigned by the matching, and the other is the type in which the assignment is blocked by a pair of agents, one of whom at least is single. In this case the assignment obtained, which satisfies the blocking pair, is preferred by the institution to the assignment μ. Formally:

Definition 5 *A matching* μ *is* $q-$ *blocked by a pair of workers* (d, e) *if*

1. $eP_d\mu(d)$, $dP_e\mu(e)$, and
2. either

a) $\mu(d) \in E$ and $\mu(e) \in D$, or

b) $\mu_{(d,e)}$ is $q-$individually rational and $\mu_{(d,e)} R_U \mu$

Definition 6 *A matching μ is $q-$stable if it is $q-$individually rational and is not $q-$blocked by any pair of workers.*

Given a matching market $M_U^q = (M; R_U, q)$, $S(M_U^q)$ denotes the set of q-stable matchings. Notice that in Femenia et al., (2008) [1] it was proved that, under the restriction of the institution's responsive preferences, the set of q-stable matchings is non-empty, i.e. $S(M_U^q) \neq \phi$. They also obtained a characterization of this set as: $S(M_U^q) = T_q(M) \cup T_{<q}(M)$.

Remark 7 *The definition of the institution's responsive preferences and of sets $T_q(M)$ and $T_{<q}(M)$ are given in detail in Appendix 1.*

3 Stability

Let us consider model $M_U^q = (M; R_U, q)$. The institution has hired q pairs of workers at most and for some reason (e.g., technological, budgetary, building-related, etc.), its hiring quota varies and a new model is obtained: $M_U^{q+1} = (M, R_U, q+1)$.

We will consider the relationship between the sets of stables in these models as well as which assignments are maintained and which are not, and which possible transformations can be applied to them in order to obtain a $q+1-$stable matching in the new model.

3.1 Matchings stables in the different models

Let us examine the relationship between sets $S(M_U^q)$ and $S(M_U^{q+1})$.

In Femenia *et al.* [5] it was shown that$T_q(M) = \underset{(t^*,s^*)}{\cup} T_q(M^{(t^*,s^*)})$. We will consider a subset K, whose elements are pairs such that, for any stable matching in the corresponding reduced model, single workers are not mutually acceptable.

Formally: Given $(t^*,\ s^*)$, let $\mu \in S(M^{(t^*,s^*)})$ and let

$$G^* = \left\{(d,e)\ :\ d \in\ D/\mu(E^{s^*}) \text{ and } e \in E/\mu(D^{t^*})\right\}$$

Note that G^* depends only on$(t^*,\ s^*)$, since those workers assigned are the same for any $\mu \in S(M^{(t^*,s^*)})$ (theorem of the single agents) and in this definition we consider the single; therefore, the definition of G^* is independent of the selected assignment μ.

$$\bar{K} = \{(t^*, s^*) \in K : \phi P_d e \text{ or } \phi P_e d \quad \forall\, (d, e) \in G^*\}$$

Let $\bar{K}^C$ be the complement of $\bar{K}$ with respect to K. Then,

$$T_q(M) = \left(\bigcup_{(t^*,s^*)\in \bar{K}} T_q(M^{(t^*,s^*)}) \right) \cup \left(\bigcup_{(t^*,s^*)\in \bar{K}^C} T_q(M^{(t^*,s^*)}) \right)$$

Let us recall that $S(M_U^{q+1}) = T_{q+1}(M) \cup T_{<q+1}(M)$. In the following proposition the set $T_{<q+1}(M)$ is characterized.

Proposition 8 *Let models M_U^q and M_U^{q+1}. Then,*

$$T_{<q}(M) \cup \left(\bigcup_{(t^*,s^*)\in \bar{K}} T_q(M^{(t^*,s^*)}) \right) = T_{<q+1}(M)$$

Proof. Let $\mu \in T_{<q}(M) \cup \left(\bigcup_{(t^*,s^*)\in \bar{K}} T_q(M^{(t^*,s^*)}) \right)$ .If $\mu \in T_{<q}(M)$,there is $(t^*,\ s^*)$ such that $\mu \in S(M^{(t^*,\ s^*)})$, $\#\mu < q$ and $\phi\ P_d\ e$ or $\phi\ P_e\ d$, for every $(d, e) \in D/\mu(E^{s^*}) \times E/\mu(D^{t^*})$; then, as $\#\mu < q + 1$ and it is stable in model $M^{(t^*,\ s^*)}$, $\mu \in T_{<q+1}(M^{(t^*,\ s^*)})$.

If $\mu \in \bigcup_{(t^*,s^*)\in \bar{K}} T_q(M^{(t^*,s^*)})$ there exists $(t^*,\ s^*) \in \bar{K}$, such that $\mu \in S(M^{(t^*,\ s^*)})$, $\#\mu < q+1$ and $\phi\ P_d\ e$ or $\phi\ P_e\ d$, for every $(d, e) \in D/\mu(E^{s^*}) \times E/\mu(D^{t^*})$, if $\#\mu = q$, as $\mu \in S(M^{(t^*,\ s^*)})$ and single workers are not mutually acceptable, then $\mu \in T_q(M^{(t^*,s^*)})$ with $(t^*,\ s^*) \in \bar{K}$. Therefore $\mu \in \bigcup_{(t^*,s^*)\in \bar{K}} T_q(M^{(t^*,s^*)})$.

If $\#\mu < q$, as$\mu \in S(M^{(t^*,\ s^*)})$, the unemployed workers are not mutually acceptable, i.e., $\mu \in T_{<q}(M)$. From previous cases $T_{<q+1}(M) \subseteq T_{<q}(M) \cup \left(\bigcup_{(t^*,s^*)\in \bar{K}} T_q(M^{(t^*,s^*)}) \right)$. ■

The next lemma provides a relation between sets $\underset{(t^*,s^*)\,\in\,\bar{K}^C}{\cup} T_q(M^{(t^*,s^*)})$ and $T_{q+1}(M)$.

Lemma 9 *If* $\mu \in \underset{(t^*,s^*)\,\in\,\bar{K}^C}{\cup} T_q(M^{(t^*,s^*)})$, *then* $\mu \notin S(M_U^{q+1})$.

Proof. Let $\mu \in \underset{(t^*,s^*)\,\in\,\bar{K}^C}{\cup} T_q(M^{(t^*,s^*)})$. Then, there is $(t^*,\ s^*) \in \bar{K}^C$such that and $\phi P_d e$ or $\phi P_e d$, for some $(d, e) \in D \times E$, with $\mu(d) = \mu(e) = \phi$. As μ is q- individually rational, then $\mu_{(d,e)}$ is $q+1$-individually rational.

As $\mu(d) = \mu(e) = \phi$ and $B_\mu \subset B_{\mu_{(d,e)}}$, by condition vi of R_U responsive preference, $\mu_{(d,e)}\ R_U\ \mu$; then, (d, e) $q+1$-blocks μ.

In addition, $\mu \notin T_{<q+1}(M)$ since there is pair of single agents who are mutually acceptable. Therefore, $\mu \notin S(M_U^{q+1})$. ■

The following theorem is an immediate consequence from previous results.

Theorem 10 *Let models* $M_U^q = (M; R_U, q)$, *and* $M_U^{q+1} = (M; R_U, q+1)$. *Then,* $S(M_u^{q+1}) \cap S(M_u^q) = T_{<q}(M) \cup \left(\underset{(t^*,s^*)\,\in\,\bar{K}}{\cup} T_q(M^{(t^*,s^*)}) \right)$.

Let $(s,t) \in N$. We will call model $M^{(s,t)}$*maximal reduced model.*

3.2 Maximal reduction model

Given a stable assignment, our objective in this section is to find a maximal reduction **of** the model, for which the assignment continues being stable, that is to say, for $\mu \in T_q(M)$ there is $(\bar{t},\ \bar{s})$, such that $\mu \in T_q(M^{(\bar{t},\bar{s})})$. However, there may be $(t^*,\ s^*) > (\bar{t},\ \bar{s})$ such that $\mu \in T_q(M^{(t^*,s^*)})$. Our aim is to find the maximal $(t^*,\ s^*)$, that is to say, the greatest pair such that $\mu \in T_q(M^{(t^*,s^*)})$ and, if we consider the reduced model obtained when an agent is added to D^{t^*} and/or an agent is added to E^{s^*}, assignment μ is not stable any more. Formally:

Definition 11 *Given $\mu \in T_q(M)$, the pair $(t^*pt,\, s^*)$ is maximal for μ if it satisfies:*

i) $\mu \in T_q(M^{(t^*,s^*)})$,

ii) either $\mu \notin T_q(M^{(t^*+1,s^*)})$ or $t^* = n$,

iii) either $\mu \notin T_q(M^{(t^*,s^*+1)})$ or $s^* = m$.

Some of the results obtained which will be useful in subsequent proofs are:

Lemma 12 *If $\mu \notin S(M^{(t^*,s^*)})$ then $\mu \notin S(M^{(\hat{t},\hat{s})})$, for every $(\hat{t},\hat{s})$ such that $(t^*,s^*) \leq (\hat{t},\hat{s})$.*

Proof. Let $\mu \notin S(M^{(t^*,s^*)})$. If μ is not individually irrational, then there is $f \in D^{t^*} \cup E^{s^*}$such that $\mu(f) \neq \phi$ and $\phi P_f \mu(f)$.

As $(t^*,s^*) \leq (\hat{t},\hat{s})$, then $D^{t^*} \cup E^{s^*} \subseteq D^{\hat{t}} \cup E^{\hat{s}}$, which implies that $f \in D^{\hat{t}} \times E^{\hat{s}}$; consequently, μ is not individually rational in $M^{(\hat{t},\hat{s})}$.

If μ is individually rational, since it is not stable, there is $(d,e) \in D^{t^*} \times E^{s^*}$, which blocks μ in model $M^{(t^*,s^*)}$.

Since $(t^*,s^*) \leq (\hat{t},\hat{s})$, then $D^{t^*} \times E^{s^*} \subseteq D^{\hat{t}} \times E^{\hat{s}}$; therefore, (d,e) blocks μ in model $M^{(\hat{t},\hat{s})}$. ■

Proposition 13 *For all $\mu \in T_q(M)$, there is maximal $(\bar{t},\bar{s})$ for μ.*

The following example shows that there may be more than one maximal pair.

Example 14 *Let $M = (D,E,P)$ with $D = \{d_1\,, d_2\,, d_3\}\,, E = \{e_1\,, e_2\,,\ \ e_3\,, e_4\}$, and let the following preferences be given by:*

$$\begin{array}{ll} P_{d_1} = e_4, e_1 & P_{e_1} = d_1 \\ P_{d_2} = e_3 & P_{e_2} = d_3 \\ P_{d_3} = e_2 & P_{e_3} = d_2 \\ & P_{e_4} = d_1 \end{array}$$

and the following individual preferences, by

$$d_1 \succ_D d_2 \succ_D d_3 \succ_D \text{ and } e_1 \succ_E e_2 \succ_E e_3 \succ_E e_4$$

The sets of stables whose assignments have cardinality 1 *are:*
$T_1(M^{(1,1)}) = T_1(M^{(1,2)}) = T_1(M^{(2,1)}) = T_1(M^{(1,3)}) = T_1(M^{(3,1)}) = T_1(M^{(2,2)}) = \{\mu_1\}$ *and* $T_1(M^{(1,4)}) = \{\mu_2\}$, *in which*

$$\mu_1 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & \phi & \phi \\ e_1 & \phi & \phi & e_2 & e_3 & e_4 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & \phi & \phi \\ e_4 & \phi & \phi & e_1 & e_3 & e_4 \end{pmatrix}$$

The sets of stables whose assignments have cardinality 2 *are:*
$T_2(M^{(2,3)}) = \{\mu_3\}$ $T_2(M^{(3,2)}) = \{\mu_4\}$ $T_2(M^{(2,4)}) = \{\mu_5\}$

$$\mu_3 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & \phi \\ e_1 & e_3 & \phi & e_2 & e_4 \end{pmatrix}, \quad \mu_4 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & \phi \\ e_1 & \phi & e_2 & e_3 & e_4 \end{pmatrix}$$

$$\mu_5 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & \phi \\ e_4 & e_3 & \phi & e_1 & e_2 \end{pmatrix}$$

The sets of stables with cardinality 3 *assignments are:*
$T_3(M^{(3,3)}) = \{\mu_6\}$ $T_3(M^{(3,4)}) = \{\mu_7\}$

$$\mu_6 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi \\ e_1 & e_3 & e_2 & e_4 \end{pmatrix}, \quad \mu_7 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi \\ e_4 & e_3 & e_2 & e_1 \end{pmatrix}$$

A quick way to visualize a maximal reduced model is to go through the rows and columns.When the assignment changes, we look for the corresponding entries. The pair whose components are these entries is a maximal pair.

$t^* \backslash s^*$	1	2	3	4
1	μ_1	μ_1	μ_1	μ_2
2	μ_1	μ_1	μ_3	μ_5
3	μ_1	μ_4	μ_6	μ_7

By going through row 1 from column 1 up to column 3, we obtain assignment μ_1, which varies to μ_2 in column 4. Pair $(1,3)$ is maximal for μ_1.

Similarly, by going through column 4, in row 1, we can see assignment μ_2; however, when we proceed to row 2, it changes to μ_5; then pair $(4,1)$ is maximal for μ_2.

By working in a similar way, we find all maximals for μ_1, which are: $(1,3)$, $(3,1)$ and $(2,2)$.

Lemma 15 *Let $\mu \in T_q(M)$, $\#D = n$ and $\#E = m$. Then, the number of maximal pairs is less or equal to* $\min\{n,m\}$.

For the proof we will consider that $n = \min\{n\,,m\}$ and we suppose that there are more than n maximal pairs. Then, for some $1 \leq t^* \leq n$, there are $s^* < s^{'}$ such that (t^*, s^*) and $(t^*, s^{'})$ are maximals. By applying the definition of maximals we get a contradiction.

Given $\mu \in T_q(M^{(t^*,s^*)})$ and $(t^{'}, s^{'})$, $(t^{''}, s^{''})$ two maximal pairs, the condition of being maximal demands that, if the first components are smaller, then second latter be bigger, i.e., if $t^{'} < t^{''}$, then $s^{''} < s^{'}$.

3.3 Procedure to determine all maximal reduced models

Given a stable assignment, we will give a procedure to determine all the maximal assignment restrictions for $\mu \in T_q(M^{(t^*,s^*)})$. For this purpose we will introduce some notations and definitions.

Let $\mu \in S(M^{(t^*,s^*)})$ such that $\#\mu = q$ and let us consider the pairs:

- $(d_{\ddot{t}}, e_{\bar{s}}) \in (D \backslash D^{t^*}) \times E^{s^*}$, which blocks μ such that, if $(d_t, e_s) \in (D \backslash D^{t^*}) \times E^{s^*}$ blocks μ, then $d_{\bar{s}} \succ_E d_s$.
- $(d_{\hat{t}}, e_{\hat{s}}) \in D \times (E \backslash E^{s^*})$, which blocks μ, such that, if $(d_t, e_s) \in D \times (E \backslash E^{s^*})$ blocks μ, then $e_{\hat{s}} \succ_E e_s$.

Notice that the blocking pair $(d_{\bar{t}}, e_{\bar{s}})$, is such that $d_{\bar{t}}$ is the first agent not assigned of D with respect to $\succ_D$, which forms a blocking pair for μ with $e_{\bar{s}} \in E^{s^*}$.

The blocking pair $(d_{\hat{t}}, e_{\hat{s}})$ is such that $e_{\hat{s}}$ is the first agent not assigned from E, with respect to $\succ_E$, which forms a blocking pair for μ with $d_{\hat{t}} \in D^{t^*}$. Note that these pairs may not exist.

Example 16 *Let $M = (D, E, P)$ be the model introduced in Example 1, and*

$$\mu_1 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & \phi & \phi \\ e_1 & \phi & \phi & e_2 & e_3 & e_4 \end{pmatrix} \in T_1(M^{(11)})$$

The pair $(d_{\bar{t}}, e_{\bar{s}})$ does not exist since e_1 is not acceptable for either d_2 or d_3 , which are single; therefore, (d_3, e_1) is not a blocking pair for μ_1, but $(d_{\hat{t}}, e_{\hat{s}}) = (d_1, e_4)$.

We now define the following

$$(d_{\tilde{t}}, e_{\tilde{s}}) = \begin{cases} (d_{\bar{t}}, e_{\bar{s}}) & \text{if there is such a pair} \\ (d_n, e_{s^*+1}) & \text{otherwise} \end{cases}$$

and

$$(d_{\check{t}}, e_{\check{s}}) = \begin{cases} (d_{\hat{t}}, e_{\hat{s}}) & \text{if there is such a pair} \\ (d_{t^*+1}, e_m) & \text{otherwise} \end{cases}$$

With these pairs of agents we can define the following sets: $\tilde{B}$ is the set of pairs of single agents in μ**,** which are mutually acceptable; the sSet $\hat{B}$ is obtained from $\tilde{B}$ by adding to it the pairs defined before.

Formally:

$$\tilde{B} = \left\{ (d_t, e_s) \in (D^{\tilde{t}} \backslash D^{t^*}) \times (E^{\check{s}} \backslash E^{s^*}) \; : \; d_t \text{ and } e_s \text{ mutually acceptable} \right\}$$

$$\hat{B} = \tilde{B} \cup \{ (d_{\tilde{t}}, e_{\tilde{s}}) \, , (d_{\check{t}}, e_{\check{s}}) \}$$

$$\bar{B} = \left\{ (d_t, e_s) \in \hat{B} \; : \; \exists \, (d_{t'}, e_{s'}) \in \hat{B} \text{ with } d_{t'} \succ_D d_t \text{ and } e_{s'} \succ_E e_s \right\}$$

Finally, we obtain the set $B = \hat{B} \backslash \bar{B}$, whose members will be the not dominated pairs. With some of the pairs from B we will obtain the maximal pairs for assignment μ.

Remark 17 *If $(d,e) \in B$ and $\mu(d) = \phi$, then $d = d_{\hat{t}}$. Symmetrically, it is also true that, given $(d,e) \in B$ and $\mu(e) = \phi$, then $e = e_{\bar{s}}$.*

We will rename the agents that form the pairs so that we can simplify the notations used in the theorems, which will allow us to find the maximal pairs for an assignment set in advance.

Let $d_{\bar{t}_1}, d_{\bar{t}_2}, \cdots, d_{\bar{t}_p}$ and $e_{\bar{s}_1}, e_{\bar{s}_2}, \cdots, e_{s_p}$ be such that

$$d_{\bar{t}1} \succ_D d_{\bar{t}2} \succ_D \succ_D d_{\bar{t}p} \text{ and } e_{\bar{s}p} \succ_E \succ_E e_{\bar{s}_1}$$

and

$$B = \left\{(d_{\bar{t}_1}, e_{\bar{s}_1}), (d_{\bar{t}_2}, e_{\bar{s}_2}), \ldots, (d_{\bar{t}_p}, e_{\bar{s}_p})\right\}$$

Note that $d_{\bar{t}_p} = d_{\tilde{t}}$ and, $e_{\bar{s}_1} = e_{\breve{s}}$.

As a consequence of the notation we have:

(I) If $(d_{\tilde{t}}, e_{\tilde{s}}) \in B$, then for every $(d_t, e_s) \in B$ if and only if $d_t \succ_D d_{\tilde{t}}$ or $d_t = d_{\tilde{t}}$.

(II) If $(d_{\breve{t}}, e_{\breve{s}}) \in B$, then for every $(d_t, e_s) \in B$ if and only if $e_s \succ_D e_{\breve{s}}$ or $e_s = e_{\breve{s}}$.

Our objective is to find a constructive method to determine all maximal pairs of an assignment. We will obtain results from set B, and they will be developed in different cases. Proofs to theorems are included in Appendix 1.

The following theorem provides conditions necessary and sufficient for pair (i,j) to be maximal if $i \neq n$ and $j \neq m$.

Theorem 18 *Let $\mu \in T_q(M^{(t^*,s^*)})$, $(i,j) \in N$ such that $i \neq n$ and $j \neq m$. The pair (i,j) is maximal for μ, if and only if, there is h such that $i = \bar{t}_h - 1$ and $j = s_{h-1} - 1$.*

Proof. See Appendix 2. ■

Now we will give the conditions necessary and sufficient for the pair (i,j) to be maximal for $i = n$ and $(d_n, e_{s^*+1}) \in \tilde{B}$ or else $(d_n, e_{s^*+1}) \notin \tilde{B}$.

Theorem 19 *Let $\mu \in T_q(M^{(t^*,s^*)})$, $(n,j) \in N$ such that $j \neq m$.*

i) *If $(d_n, e_{s^*+1}) \in \tilde{B}$, the pair (n,j) is maximal for μ if and only if $(d_n, e_{s^*+1}) \in B$ and $j = \overline{s}_p - 1$.*

ii) *If $(d_n, e_{s^*+1}) \notin \tilde{B}$, the pair (n,j) is maximal for μ if and only if $(d_n, e_{s^*+1}) \in B$ and $j = \overline{s}_{p-1} - 1$.*

Proof. See Appendix 2. ■

In order to complete the constructive method, Theorem 6, which is the symmetrical result to that obtained in Theorem 5, completes the characterization.

Theorem 20 *Let $\mu \in T_q(M^{(t^*,s^*)})$, $(i,m) \in N$ such that $i \neq n$.*

i) If $(d_{t^*+1}, e_m) \in \tilde{B}$, the pair (i,m) is maximal for μ if and only if $(d_{t^*+1}, e_m) \in B$ and $i = \bar{t}_1 - 1$.

ii) If $(d_{t^*+1}, e_m) \notin \tilde{B}$, the pair (i,m) is maximal for μ if and only if $(d_{t^*+1}, e_m) \in B$ and $i = \bar{t}_2 - 1$.

Proof. See Appendix 2. ■

In the model of Example 1 we find that all maximals for μ_1 by direct verification. Now we will see, in the same model, how to find maximals by using the results obtained.

Example 21 (Cont.) *Let*

$$\mu_1 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & \phi & \phi \\ e_1 & \phi & \phi & e_2 & e_3 & e_4 \end{pmatrix} \in T_1(M^{(11)})$$

$(d_{\bar{t}}, e_{\bar{s}})$ does not exist since e_1 is not acceptable for either d_2 and $\mu_1(d_3) = \phi$, and (d_3, e_1) is not a blocking pair for μ_1; but $(d_{\check{t}}, e_{\check{s}}) = (d_3, e_2)$.

As (d_1, e_4) blocks μ_1 then $(d_{\hat{t}}, e_{\hat{s}}) = (d_1, e_4)$. By means of these pairs lets us find sets:

$$\tilde{B} = \{(d_2, e_3), (d_3, e_2)\}$$

$$\hat{B} = \{(d_2, e_3), (d_3, e_2)\} \cup \{(d_2, e_3), (d_1, e_4)\}$$
$$\bar{B} = \phi$$

Therefore, $B = \{(d_2, e_3), (d_3, e_2), (d_1, e_4)\}$. *As* $d_1 \succ_D d_2 \succ d_3$, *let us rename the agents by:* $d_{\bar{t}_1} = d_1$, $d_{\bar{t}_2} = d_2$, $d_{\bar{t}_3} = d_3$; *as a consequence,* $e_{\bar{s}_p} = e_2$, $e_{\bar{s}_2} = e_3$ *and* $e_{\bar{s}_1} = e_4$. *We can notice that* $d_{\bar{t}_p} = d_{\check{t}} = d_3$ *and* $e_{\bar{s}_p} = e_{\bar{s}_1} = e_4$.

Case 1 *Find maximal pair* (i, j) *when* $i \neq 3$ *and* $j \neq 4$.

For this purpose, we will apply Theorem 4 and we will consider the following pairs in B.

- (d_1, e_4) *and* (d_2, e_3). *We find that* $(1, 3)$ *is maximal.*
- (d_2, e_3) *and* (d_3, e_2). *We have that* $(2, 2)$ *is maximal.*

Case 2 *Find maximal pair* (i, j) *when* $i = 3$. *We will apply Theorem 5.*

As $(d_3, e_2) \in B$, *and* d_3 *and* e_2 *are mutually acceptable , by (i)* $(3, 1)$ *is maximal.*

Case 3 *Find maximal pair* (i, j) *when* $j = 4$. *We will apply Theorem 6.*

As $(d_2, e_4) \notin B$, *there are no maximal pairs that take this form. Then, the maximal pairs for* μ_1 *are:* $(1, 3)$, $(2, 2)$ *and* $(3, 1)$.

3.4 An algorithm to calculate the stables in M_u^{q+1}

Given a model $M = (D, E, P)$, by theorem of the single agents we can conclude that all the matchings in the model have the same cardinality.

Let $\mu \in S(M)$. Note that, if $\#\mu < q + 1$, both sets of stables coincide: $S(M_U^q)$=$S(M_U^{q+1})$. If $\#\mu \geq q + 1$, we only have some of the stable assignments of the new model, which were analysed in 3.1 (We saw that with some of the matchings of $S(M_U^q)$ we can characterize $T_{<q+1}(M)$. In this section we will design an algorithm to determine the other set, i.e. $T_{q+1}(M)$.

Notice that, if $\mu \in \underset{(t^*,s^*) \in \bar{K}^C}{\cup} T_q(M^{(t^*,s^*)})$, by Lemma 3 $\mu \notin T_{q+1}(M)$.

We will consider how we can transform this matching into an assignment in this set of stables.

Let $\mu \in T_q(M^{(\bar{t},\bar{s})})$, with $(\bar{t},\bar{s}) \in \bar{K}^C$, then $\mu \in S(M^{(\bar{t},\bar{s})})$, $\#\mu = q$. Notice that, by Proposition 5, there is at least one maximal pair for μ. We can suppose that $(\bar{t},\bar{s})$ is maximal. By the definition of maximal pair, $\mu \notin S(M^{(\bar{t}+1,\bar{s})})$ and $\mu \notin S(M^{(\bar{t},\bar{s}+1)})$. Notice that, if $\bar{t} = n$ or $\bar{s} = m$, it is possible to consider only one of the sets of stables. Without loss of generality and for the sake of notational simplicity, we will consider only the case in which $\bar{t} \neq n$ and $\bar{s} \neq m$.

Let us consider the model $M^{(\bar{t}+1,\bar{s})}$. We will obtain the set $S(M^{(\bar{t}+1,\bar{s})})$. Let $\bar{\mu} \in S(M^{(\bar{t}+1,\bar{s})})$. By Lemma A1 in Appendix 1, $\#\bar{\mu} = q$ or $\#\bar{\mu} = q+1$.

If $\#\bar{\mu} = q+1$, all the assignments in $S(M^{(\bar{t}+1,\bar{s})})$ have cardinality $q+1$, then $S(M^{(\bar{t}+1,\bar{s})}) \subseteq T_{q+1}(M)$.

Let $\bar{\mu} \in S(M^{(\bar{t}+1,\bar{s})})$ and let one of its maximal pairs be (t^*, s^*). By the definition of maximal pair, $\bar{\mu} \notin S(M^{(t^*+1,s^*)})$ and $\bar{\mu} \notin S(M^{(t^*,s^*+1)})$.

Let $\tilde{\mu} \in S(M^{(t^*+1,s^*)})$. By Lemma A1, Appendix 1, $\#\tilde{\mu} = q+1$ or $\#\tilde{\mu} = q+2$.

If $\#\tilde{\mu} = q+1$, we will repeat the procedure applied for $\bar{\mu}$. If $\#\tilde{\mu} = q+2$, we do not consider it.

The procedure has to be repeated for every maximal pair of μ, $\bar{\mu}$ and $\tilde{\mu}$ too.

If $\#\bar{\mu} = q$, as $\bar{\mu} \in T_q(M^{(\bar{t}+1,\bar{s})})$, there is maximal $(\hat{t},\hat{s})$. We consider models $M^{(\hat{t}+1,\hat{s})}$ and $M^{(\hat{t},\hat{s}+1)}$. We find again the set of stables and we continue until we find assignments whose cardinals are $q+1$.

We apply the same procedure for model $M^{(t^*,s^*+1)}$. Finally, we prove that by means of it, we can find $T_{q+1}(M)$.

A formal description of the methodology followed can be consulted in Appendix 3.

In our next example we can see its application to determine assignments with cardinality q.

Example 22 *Let* $M_U^2 = (M, R_U, 2)$ *with* $D = \{d_1, d_2\}, E = \{e_1, e_2\}$ *, the* *following preferences :*

$$\begin{array}{ll} P_{d_1} = e_2, e_1 & P_{e_1} = d_2, d_1 \\ P_{d_2} = e_1 & P_{e_2} = d_1, d_2 \end{array}$$

and the following individual preferences:

$$d_1 \succ_D d_2 \quad and \quad e_1 \succ_E e_2$$

If we calculate $S(M_U^1)$*, by definition we obtain:* $S(M_U^1) = \{\mu_1, \mu_2, \mu_3\}$*, where*

$$\mu_1 = \begin{pmatrix} d_1 & d_2 & \phi \\ e_1 & \phi & e_2 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} d_1 & d_2 & \phi \\ \phi & e_1 & e_2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} d_1 & d_2 & \phi \\ e_2 & \phi & e_1 \end{pmatrix}$$

$T_1(M^{(1,1)}) = \{\mu_1\}$ $T_1(M^{(2,1)}) = \{\mu_2\}$ $T_1(M^{(1,2)}) = \{\mu_3\}$.

By applying $S(M_U^2)$*, by definition we find:* $S(M_U^2) = \{\mu_4\}$*, where* $\mu_1 = \begin{pmatrix} d_1 & d_2 \\ e_2 & e_1 \end{pmatrix}$ *and* $T_2(M^{(2,2)}) = \{\mu_4\}$.

We apply the procedure applied above, starting from $S(M_U^1)$.

Let $\mu_1 \in T_1(M^{(1,1)})$*, but* $(1,1) \notin \bar{K}^C$*, given that* d_2 *and* e_2 *are not mutually acceptable, so we begin with* μ_1.

Let $\mu_2 \in T_1(M^{(2,1)})$*. We find the maximals for* μ_2 *, which is the pair* $(2,1)$*. We consider model* $M^{(2,2)}$ *and obtain its set of stables:* $S(M^{(2,2)}) = \{\mu_4\}$ *with* $\#\mu_4 = 2$.

Given $\mu_3 \in T_1(M^{(1,2)})$*, we find the maximals for* μ_3 *in a similar way and we get* $S(M^{(2,2)}) = \{\mu_4\}$.

4 Conclusions

In this research work, given a matching model with restriction capacity, we obtain another model by increasing the hiring quota in the original model. First we make an analysis of the assignments that are kept in both models and of those that are not. Then we define the concepts of a maximal pair and maximal reduced model, and we describe a method to calculate the maximal pairs of a q-stable matching to complete the new set of $q+1$-stable. Finally, we develop a procedure by using maximal pairs by means of which we can obtain the whole set of $q+1$-stables.

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Appendix 1

The restriction of M

From now on, we will denote $F \in \{D, E\}$ and $F^c \in \{D, E\}$ such that $\{F, F^c\} = \{D, E\}$, and $f \in F$ will denote a generic worker.

Given $F^{'} \subseteq F$, we denote the restriction of P_F to $F^{'}$ by $P_{|F'}$. Given $M = (F, F^C, P)$, we denote the restriction of M to $F^{'}$by $M_{F'} = (F^{'}, F^C, P_{|F'}, P_{F^C})$. For the sake of simplicity we denote $M_{F'} = (F^{'}, F^C, P)$, where we have to understand that $P = (P_{|F'}, P_{F^C})$.

Lemma 23 (Femenia, Marí, Oviedo and Neme, 2010) *Given $M = (D, E, P)$ and $F^{'} \subseteq F$, let μ and μ be the stable matchings for M and $M_{F'}$ respectively. Then $\#\mu^{'} \leq \#\mu \leq \#\mu^{'} + \#(F \backslash F^{'})$.*

The institution's responsive preference

Given a matching market M_U and a quota $q \leq \min\{n, m\}$, we denote by $S(M_U^q)$ the set of all $q-$stable matchings. We will assume that the institution has an individual preference $\succ_D$over the set $D \cup \phi$ and an individual preference $\succ_E$over the set $E \cup \phi$ and its preference over matchings are directly connected with its preferences over workers. An institution's preference is called responsive to its individual preferences if, for any matching that differs in only one worker, the institution prefers the matching that has the most preferable worker according to the individual preferences.

In order to formalize the institution's responsive preference, we introduce the notations that follow.

For every matching μ, consider $B_\mu = \{(d, e) \in D \times E : \mu(d) = e\}$. For every $f \in D \cup E$: $\mu^{(d,e)}(f) = \begin{cases} \phi & \text{if } f \notin \{d, e\} \\ d & \text{if } f = e \end{cases}$

Notice that $\mu^{(d,e)} = \mu^{\emptyset}_{(d,e)}$.

Definition 24 *A preference relation R_U is a responsive extension of preferences $\succ_D$and $\succ_E$over $D \cup \{\phi\}$ and $E \cup \{\phi\}$ respectively, such that it satisfies the following conditions:*

i) $\mu^{(d,e)} P_U \mu^{\phi}$ *if and only if* $d \succ_D \phi$ *and* $e \succ_E \phi$.

ii) $\mu P_U \mu^{\phi}$ *if and only if* $\mu^{(d,e)} P_U \mu^{\phi}$ *for every* $(d,e) \in B_{\mu}$.

iii) $\mu^{(d,e)} P_U \mu^{(d,e^{'})}$ *if and only if* $e \succ_E e^{'}$.

iv) $\mu^{(d,e)} P_U \mu^{(d^{'},e)}$ *if and only if* $d \succ_D d^{'}$.

v) *For every* $\mu,\ \mu^{'} \in M$ *such that* $\#\mu = \#\mu^{'}$ *and* $B_{\mu} = B_{\mu^{'}} \backslash \{(d^{'},e^{'})\} \cup \{(d,e)\}$*:* $\mu P_U \mu^{'}$ *if and only if* $\mu^{(d,e)} P_U \mu^{(d^{'},e^{'})}$.

vi) *For every* $\mu,\ \mu^{'} \in M$ *such that* $B_{\mu^{'}} \subset B_{\mu}$ *and* $\mu P_U \mu^{\phi}$, *then* $\mu P_U \mu^{'}$.

vii) *For every* $\mu,\ \mu^{'} \in M$ *such that* $\mu(E) = \mu^{'}(E)$ *and* $\mu(D) = \mu^{'}(D)$, *then* $\mu\ I_U \mu^{'}$.

We consider a preference R_U to be responsive if there are two individual preferences $\succ_D$and $\succ_E$over $D \cup \phi$ and $E \cup \phi$ respectively, such that R_U is a responsive extension.

Remark 25 *Given two preferences* $\succ_D$*and* $\succ_E$, *over* $D \cup \phi$ *and* $E \cup \phi$ *respectively, we can construct a responsive preference relation* R_U *over the set of all matchings* M *; moreover, this extension is not unique.*

The sets $T_q(M)$ and $T_{<q}(M)$

Now we will consider the model M_U^q, where R_U is a responsive preference. Without loss of generality and in order to avoid the addition of notational complexity to the model M_U^q, we assume that all the agents of sets D and E are acceptable for the institution, i.e. for every $d \in D$ and $e \in E$, we have that $d \succ_D \phi$ and $e \succ_E \phi$.

For every $t \in \mathbb{N}$, we can define the following subset $F^t \subseteq F$ such that $\#F^t = t$, and for every $f \in F^t$ and $f^{'} \notin F^t$ we have that $f \succ_F f^{'}$. Note that $F^1 \subseteq F^2 \subseteq \cdots \subseteq F^l = F$, where $\#F = l$.

Given sets $d = \{1, 2, ..., \#D\}$ and $e = \{1, 2, ..., \#E\}$; for every $(t_1, t_2) \in d \times e$, we denote $M^{(t_1,t_2)}$, the restriction of M to D^{t_1} and E^{t_2}, i.e., $M^{(t_1,t_2)} = \left(D^{t_1}, E^{t_2}, P\right)$.

Given $(t_1, t_2) \in d \times e$, q, and the following sets of matchings:

$$T_q(M^{(t^*,\,s^*)}) = \begin{cases} S(M^{(t^*,\,s^*)}) & \text{if } \#\mu = q \text{ for every } \mu \in S(M^{(t^*,\,s^*)}) \\ \phi & \text{otherwise} \end{cases}$$

and $T_q(M) = \left\{\mu : \exists (t^*, s^*) \text{ such that } \mu \in T_q(M^{(t^*,\,s^*)})\right\}$.

Proposition 26 *Given $M_U = (M, R_U), (t^*, s^*) \in d \times e$, there exists $K \subseteq d \times e$, such that*

$$T_q(M) = \bigcup_{(t^*,\,s^*) \in K} T_q(M^{(t^*,\,s^*)})$$

Given $(t_1, t_2) \in d \times e$, q and the following sets of stable matchings:

$$T_{<q}(M^{(t^*,\,s^*)}) = \; \left\{\mu \in S(M^{(t^*,\,s^*)}) : \#\mu < q, \text{ either } \phi P_e d \text{ or } \phi P_d e \right. \\ \left. \text{for every } d \in D \backslash \mu(E^{s*}) \text{ and } e \in E \backslash \mu(D^{t^*})\right\}$$

And $T_{<q}(M) = \left\{\mu : \exists (t^*, s^*) \text{ such that } \mu \in T_{<q}(M^{(t^*,\,s^*)})\right\}$.

Proposition 27 *Given $M_U = (M, R_U), (t^*, s^*) \in d \times e$, there exists $K \subseteq d \times e$, such that*

$$T_{<q}(M) = \bigcup_{(t^*,\,s^*) \in \hat{K}} T_{<q}(M^{(t^*,\,s^*)})$$

Remark 28 *The sets K and $\hat{K}$ on the previous propositions are given by: $K = \{(t^*, s^*) \in N : \forall\ (t_1', t_2') \neq (t_1, t_2),\ t_1' \leq t_1$ and $t_2' \leq t_2$ such that $T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t_1',t_2')}) = \emptyset\}$ and $\hat{K} = \{(t_1, t_2) \in N : \forall\ (t_1', t_2') \neq (t_1, t_2),\ t_1' \leq t_1$ and $t_2' \leq t_2$ such that $T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t_1',t_2')}) = \emptyset\}$.*

Now, we are going to present the following results which state that the set of q-stable matching are non-empty.

Theorem 29 *If $M_U^q, = (M, R_U, q)$ is a matching market, then $S(M_U^q) \neq \phi$.*

The following theorem is a complete characterization of the q-stable sets $S(M_U^q)$.

Theorem 30 *If $M_U^q, = (M, R_U, q)$ is a matching market, then*

$$S(M_U^q) = T_q(M) \cup T_{<q}(M).$$

Appendix 2

Theorem 31 *Let $\mu \in T_q(M^{(t^*,s^*)})$, $(i,j) \in N$ such that $i \neq n$ and $j \neq m$. The pair (i,j) is maximal for μ, if and only if, there exists h such that $i = \bar{t}_h - 1$ and $j = s_{h-1} - 1$.*

Proof. *First we will prove the condition of being sufficient.*

Let h such that $i = \bar{t}_h - 1$ and $j = \bar{s}_{h-1} - 1$. Let us prove that the pair (i,j) is maximal for μ. Let us consider model $M^{(\bar{t}_h-1,\bar{s}_{h-1}-1)}$, in which $D^{t*} \subseteq D^{\bar{t}_h-1}$ and $E^{s^*} \subseteq E^{\bar{s}_{h-1}-1}$ and let us prove that $\mu \in T_q(M^{(\bar{t}_h-1,\bar{s}_{h-1}-1)})$. Suppose that $\mu \notin S(M^{(\bar{t}_h-1,\bar{s}_{h-1}-1)})$. As $\mu \in T_q(M^{(t^*,s^*)})$, we can say that the agents in D who have been assigned another agent are in D^{t^*} and that the agents in E who have been assigned some agent are in E^{s^*}. Then, μ is individually rational in $M^{(\bar{t}_h-1,\bar{s}_{h-1}-1)}$ and, as μ is not stable, there is a pair $(\tilde{d},\tilde{e}) \in D^{\bar{t}_h-1} \times E^{\bar{t}_{h-1}-1}$ which blocks μ, i.e. $\tilde{e}\ P_{\tilde{d}}\ \mu(\tilde{d})$ and $\tilde{d}\ P_{\tilde{e}}\ \mu(\tilde{e})$.

Without loss of generality, we can consider that for every (d,e) which blocks μ in $M^{(\bar{t}_h-1,\bar{s}_{h-1}-1)}$, $\tilde{d} \succ_D d$, that is to say, $(\tilde{d},\tilde{e}) \in B$.

Indeed, suppose that $(\tilde{d},\tilde{e}) \notin B$. As $\tilde{d}$ and $\tilde{e}$ are mutually acceptable agents and $(\tilde{d},\tilde{e}) \in \hat{B}$, then, $(\tilde{d},\tilde{e}) \in \bar{B}$. Therefore, there are $(d',e') \in \hat{B}$ such that $d' \succ_D \tilde{d}$ and $e' \succ_E \tilde{e}$, which contradicts the fact that the pair considered $(\tilde{d},\tilde{e})$ is such that $\tilde{d} \succ_D d'$.

As $D^{t*} \subseteq D^{\bar{t}_h-1}$, the following cases may occur:

Case 1 *$\tilde{d} \succ_D d_{t^*}$ or $\tilde{d} = d_{t^*}$. Since $(\tilde{d},\tilde{e})$ blocks μ and $\tilde{d} \in D^{t^*}$, the defintion of $e_{\breve{s}}$ implies that $e_{\breve{s}} \succ_E \tilde{e}$ or $\tilde{e} = e_{\breve{s}}$. In addition, $(d_{\bar{t}_{h-1}}, e_{\ddot{s}_{h-1}}) \in$*

B and $e_{\dddot{s}_{h-1}} \succ_E e_{\breve{s}}$ or $e_{\dddot{s}_{h-1}} = e_{\breve{s}}$; therefore $e_{\dddot{s}_{h-1}} \succ_E \tilde{e}$ or $e_{\dddot{s}_{h-1}} = \tilde{e}$. By definition of $E^{s_{h-1}-1}$, $\tilde{e} \notin E^{s_{h-1}-1}$. Then, $(\tilde{d}, \tilde{e})$ is not a blocking pair in model $M^{(\bar{t}_h, \bar{s}_{h-1}-1)}$.

Case 2 *$d_{t^*} \succ_D \tilde{d} \succ_D d_{\bar{t}_{h-1}}$ or $\tilde{d} = d_{\bar{t}_{h-1}}$. Since $(\tilde{d}, \tilde{e}) \in B$; therefore, $e_{\dddot{s}_{h-1}} \succ_E \tilde{e}$ or $e_{\dddot{s}_{h-1}} = \tilde{e}$, which implies that $(\tilde{d}, \tilde{e}) \notin D^{\bar{t}_h - 1} \times E^{\bar{t}_{h-1}-1}$ and contradicts that $(\tilde{d}, \tilde{e})$ blocks μ in $M^{(\bar{t}_h, \bar{s}_{h-1}-1)}$.*

Case 3 *$d_{\bar{t}_{h-1}} \succ_D \tilde{d} \succ_D d_{\bar{t}_h}$. Since $(d_{\bar{t}_{h-1}}, e_{\dddot{s}_{h-1}})$ and $(d_{\bar{t}_h}, e_{\dddot{s}_h}) \in B$ are such that $d_{\bar{t}_{h-1}} \succ_D d_{\bar{t}_h}$ and there is no $(d, e) \in B$ such that $d_{\bar{t}_{h-1}} \succ_D d \succ_D d_{\bar{t}_h}$. This case cannot occur since $(\tilde{d}, \tilde{e}) \in B$. Then, $\mu \in S(M^{(\bar{t}_h - 1, \bar{s}_{h-1}-1)})$.*

Now Let us consider model $M^{(\bar{t}_h, \bar{s}_{h-1}-1)}$. As $(d_{\bar{t}_{h-1}}, e_{\dddot{s}_{h-1}})$ and $(d_{\bar{t}_h}, e_{\dddot{s}_h}) \in B = \hat{B} \backslash \bar{B}$, with $d_{\bar{t}_{h-1}} \succ_D d_{\bar{t}_h}$. Then, $e_{\bar{s}_h} \succ_E e_{\bar{s}_{h-1}}$. Therefore, $e_{\bar{s}_h} \in E^{\bar{s}_{h-1}-1}$, and, as $d_{\bar{t}_h} \in D^{\bar{t}_h - 1}$, $(d_{\bar{t}_h}, e_{\dddot{s}_h})$ blocks μ in model $M^{(\bar{t}_h, \bar{s}_{h-1}-1)}$. Then, $\mu \notin S(M^{(\bar{t}_h, \bar{s}_{h-1}-1)})$.

Let us take model $M^{(\bar{t}_h - 1, \bar{s}_{h-1})}$. As $(d_{\bar{t}_{h-1}}, e_{\dddot{s}_{h-1}})$ and $(d_{\bar{t}_h}, e_{\dddot{s}_h}) \in B$, with $d_{\bar{t}_{h-1}} \succ_D d_{\bar{t}_h}$, then, $d_{\bar{t}_{h-1}} \in D^{\bar{t}_h - 1}$ and $e_{\bar{s}_{h-1}} \in E^{\bar{s}_{h-1}-1}$. The pair $(d_{\bar{t}_{h-1}}, e_{\dddot{s}_{h-1}})$ blocks μ in this model. Therefore, $\mu \notin S(M^{(\bar{t}_h - 1, \bar{s}_{h-1})})$.

Now, we will prove the condition of necessity.

Let $(i, j) \in N$ with $i \neq n$ and $j \neq m$ be maximal for μ. Let us prove that there is h such that $i = \bar{t}_h - 1$ and $j = s_{h-1} - 1$. (i, j) is maximal for μ; then, $\mu \in S(M^{(i,j)})$, $\mu \notin S(M^{(i+1,j)})$ and $\mu \notin S(M^{(i,j+1)})$. Since $\mu \in S(M^{(i,j)})$, $\mu \notin S(M^{(i+1,j)})$, there is $e_k \in E^j$ such that $(d_{i+1}, e_k) \in B$. Since $\mu \in S(M^{(i,j)})$, $\mu \notin S(M^{(i,j+1)})$, there is $d_l \in D^i$ such that $(d_l, e_{j+1}) \in B$.

Now, owing to the way we have symbolized the pairs in B, there are h' and h'' such that $(d_l, e_{j+1}) = (d_{\bar{t}_{h''}}, e_{\bar{s}_{h''}})$ and $(d_{i+1}, e_k) = (d_{\bar{t}_{h'}}, e_{\bar{s}_{h'}})$, with $e_k \succ_E e_{j+1}$ and $d_l \succ_D d_{i+1}$; that is to say, $e_{\bar{s}_{h'}} \succ_E e_{j+1}$ and $d_{\bar{t}_{h''}} \succ_D d_{i+1}$.

As $e_{\bar{s}_{h'}} \succ_E e_{j+1} = e_{\bar{s}_{h''}}$, then $d_{\bar{t}_{h''}} \succ_D d_{\bar{t}_{h'}}$. It remains to prove that there is no $(d, e) \in B$ such that $d_{\bar{t}_{h''}} \succ_D d \succ_D d_{\bar{t}_{h'}}$. Let us suppose that there exists $(d, e) \in B$ such that $d_{\bar{t}_{h''}} \succ_D d \succ_D d_{\bar{t}_{h'}}$. Given that $(d, e) \in B$, then

$e_{\bar{s}_{h'}} \succ_E e \succ_E e_{\bar{s}_{h''}}$. *That is to say,* $d_l \succ_D d \succ_D d_{i+1}$ *and* $e_k \succ_E e \succ_E e_{j+1}$. *Then* $(d, e) \in D^i \times E^j$, *which contradicts* $\mu \in S(M^{(i,j)})$. ■

Theorem 32 *Let* $\mu \in T_q(M^{(t^*,s^*)})$, $(n, j) \in N$ *such that* $j \neq m$.

i) *If* $(d_n, e_{s^*+1}) \in \tilde{B}$, *the pair* (n, j) *is maximal for* μ *if and only if* $(d_n, e_{s^*+1}) \in B$ *and* $j = \bar{s}_{p-1} - 1$.

ii) *If* $(d_n, e_{s^*+1}) \notin \tilde{B}$, *the pair* (n, j) *is maximal for* μ *if and only if* $(d_n, e_{s^*+1}) \in B$ *and* $j = \bar{s}_{p-1} - 1$.

Proof. *Let us prove (i) first. For the necessary condition, let* (n, j) *be maximal for* μ, *then there is no* $(d, e) \in D \times E^j$ *blocking* μ. *As* $E^{s^*} \subseteq E^j$, *the is no* $(d, e) \in D \times E^{s^*}$ *blocking* μ. *Then, the pair* $(d_{\bar{t}}, e_{\bar{s}})$ *does not exist. Then,* $(d_n, e_{s^*+1}) = (d_{\tilde{t}}, e_{\tilde{s}}) \in \tilde{B}$.
Let us prove now that $(d_n, e_{s^*+1}) \in B$. *For this purpose, let us suppose that* $(d_n, e_{s^*+1}) \notin B$; *then there is* $(d, e) \in \hat{B}$ *such that* $d \succ_D d_n$ *and* $e \succ_E e_{s^*+1}$. *Therefore,* $(d, e) \in D \times E^{s^*}$ *blocks* μ, *which contradicts* $\mu \in S(M^{(n,s^*)})$. *As a consequence,* $(d_n, e_{s^*+1}) \in B$. *In addition, as* (n, j) *is maximal, there is* $d_l \in D^i$, *such that* $(d_l, e_{j+1}) \in B$.

We now prove that $(d_l, e_{j+1}) = (d_n, e_{s^*+1}) = (d_{\tilde{t}}, e_{\tilde{s}})$. *Since* $(d_{\tilde{t}}, e_{\tilde{s}}) = (d_{\bar{t}_p}, e_{\bar{s}_p})$, *let us verify that* $(d_l, e_{j+1}) = (d_{\bar{t}_p}, e_{\bar{s}_p})$. *That is to say,* $e_{j+1} = e_{\bar{s}_p}$, *from which* $j = \bar{s}_{p-1} - 1$. *If* $e_{\bar{s}_p} \succ_E e_{j+1}$, *then* $e_{\bar{s}_p} \succ_E e_j$ *or* $e_{\bar{s}_p} = e_j$. *Then* $e_{\bar{s}_p} \in E^j$ *and therefore,* $(d_{\bar{t}_p}, e_{\bar{s}_p}) \in D \times E^j$, *and as* $(d_{\bar{t}_p}, e_{\bar{s}_p}) = (d_n, e_{s^*+1}) \in B$, *these agents are mutually acceptable and* $(d_{\bar{t}_p}, e_{\bar{s}_p})$ *blocks* μ *in* $M^{(n,j)}$. *This contradiction arises for having supposed that* $e_{\bar{s}_p} \succ_E e_{j+1}$. *Then* $e_{j+1} \succ_E e_{\bar{s}_p}$ *or* $e_{j+1} = e_{\bar{s}_p}$. *We know that* $(d_l, e_{j+1}) \in B$ *and* $(d_{\bar{t}_p}, e_{\bar{s}_p}) = (d_n, e_{s^*+1}) \in B$, *in which* $d_l \succ_D d_{\bar{t}_p}$ *or* $d_l = d_{\bar{t}_p}$. *If* $e_{j+1} \succ_E e_{\bar{s}_p}$, *the pair* $(d_{\bar{t}_p}, e_{\bar{s}_p}) \in \bar{B}$, *which is a contradiction, therefore* $e_{j+1} = e_{\bar{s}_p}$.

For the sufficient condition, since $(d_n, e_{s^*+1}) \in B$, *then* d_n *and* e_{s^*+1} *are mutually acceptable. We first prove that* (n, s^*) *is maximal for* μ. *Let us consider model* $M^{(n,s^*)}$ *and prove that* $\mu \in T_q(M^{(n,s^*)})$. *Since* $\#\mu = q$ *, it remains to prove that* $\mu \in S(M^{(n,s^*)})$. *Suppose that* $\mu \notin S(M^{(n,s^*)})$. *As* μ *is stable in* $M^{(t^*,s^*)}$, μ *is individually rational in the model . Then,* *there is a pair* $(\tilde{d},\tilde{e}) \in D \times E^{s^*}$ *which blocks* μ *in this model. Without loss of generality we can consider that, for every* (d,e) *which blocks* μ *in* $M^{(n,s^*)}$, $\tilde{e} \succ_E e$. *This implies that* $(\tilde{d},\tilde{e}) \in B$ *and* $\tilde{e} \succ_E e_{s^*}$ *or* $\tilde{e} = e_{s^*}$. *As* $(\tilde{d},\tilde{e}) \in B$ *and* $\tilde{d} \succ_D d_n$, *then* $e_{s^*+1} \succ_E \tilde{e}$. *Therefore* $e_{s^*+1} \in E^{s^*}$, *which contradicts the definition of* E^{s^*}, *where* $\mu \in S(M^{(n,s^*)})$. *Next, consider the model* $M^{(n,s^*+1)}$. *Since* $(d_n, e_{s^*+1}) \in D \times E^{s^*+1}$, *agents* d_n *and* e_{s^*+1} *are mutually acceptable. Then,* (d_n, e_{s^*+1}) *blocks* μ *in* $M^{(n,s^*+1)}$. *As a consequence,* $\mu \notin S(M^{(n,s^*+1)})$.

We now prove (ii). For this purpose, let us prove the condition of being sufficient. As $(d_n, e_{s^*+1}) \in B$, *then* $d_n = d_{\bar{t}_p}$ *and* $(d_n, e_{s^*+1}) = (d_{\bar{t}_p}, e_{\bar{s}_p})$. *Let* $(d_{\bar{t}_{p-1}}, e_{\bar{s}_{p-1}}) \in B$ *and let us prove that* $(n, \bar{s}_{p-1}-1)$ *is maximal .*

Let us consider the model $M^{(n,\bar{s}_{p-1}-1)}$, *in which* $D^{t^*} \subseteq D^n, E^{s^*} \subseteq E^{\bar{s}p-1^{-1}}$ *and let us prove that* $\mu \in T_q(M^{(n,\bar{s}_{p-1}-1)})^{\mu \in T_q(M^{(n,\bar{s}_{p-1}-1)})}$. *As* $\#\mu = q$, *it remains to prove that* $\mu \in S(M^{(n,\bar{s}_{p-1}-1)})$. *Suppose that* $\mu \notin S(M^{(n,\bar{s}_{p-1}-1)})$. *Since* μ *is stable in* $M^{(t^*,s^*)}$, μ *is individually rational in* $M^{(n,\bar{s}_{p-1}-1)}$. *Then there is a pair* $(\tilde{d},\tilde{e}) \in D \times E^{\bar{s}_{p-1}-1}$ *which blocks* μ *in this model.*

Without loss of generality we can consider that for every (d,e) *which blocks* μ *in* $M^{(n,\bar{s}_{p-1}-1)}$, $\tilde{e} \succ_E e$. *This implies that* $(\tilde{d},\tilde{e}) \in B$.

Indeed, suppose that $(\tilde{d},\tilde{e}) \notin B$. *As* $\tilde{d}$ *and* $\tilde{e}$ *are mutually acceptable agents and* $(\tilde{d},\tilde{e}) \in \hat{B}$, *then,* $(\tilde{d},\tilde{e}) \in \bar{B}$. *Therefore, there is* $(d', e') \in \hat{B}$ *such that* $d' \succ_D \tilde{d}$ *and* $e' \succ_E \tilde{e}$, *which contradicts the fact that the pair considered* $(\tilde{d},\tilde{e})$ *is such that* $\tilde{e} \succ_E e'$. *As* $(d_n, e_{s^*+1}) \in B$, *then* $e_{\bar{s}_p} \succ_E \tilde{e} \succ_E e_{\bar{s}_{p-1}}$, *which is a contradiction, since in the individual preferences* $\succ_E$*between agents* $e_{\bar{s}_p}$ *and* $e_{\bar{s}_{p-1}}$, *there is no other agent forming a pair in* B. *Then,* $\mu \in S(M^{(n,\bar{s}_{p-1}-1)})$.

Let us consider the model $M^{(n,\bar{s}_{p-1})}$. *Since* $(d_{\bar{t}_{p-1}}, e_{\bar{s}_{p-1}}) \in D \times E^{\bar{s}_{p-1}}$ *and* $(d_{\bar{t}_{p-1}}, e_{\bar{s}_{p-1}}) \in B$, *the latter blocks* μ *in* $M^{(n,\bar{s}_{p-1})}$; *therefore* $\mu \notin S(M^{(n,\bar{s}_{p-1})})$.

Now, let us prove now the necessary condition. Note that $E^{s^*} \subseteq E^j$. *As* (n,j) *is maximal for* μ *,then there is no* $d \in D^n = D$ *and* $e \in E^{s^*}$*mutually acceptable. Then by the definition of* $(d_{\tilde{t}}, e_{\tilde{s}})$ *we have that* $(d_n, e_{s^*+1}) = (d_{\bar{t}_p}, e_{\bar{s}_p}) \in B$. *As* (n,j) *is maximal for* μ, *then* $\mu \in S(M^{(n,j)})$ *and* $\mu \notin$

$S(M^{(n,j+1)})$. There is d_t such that $(d_t, e_{j+1}) \in B$. As$(d_n, e_{s^*+1}) \notin \tilde{B}$, this pair does not block μ and then, $(d_t, e_{j+1}) \neq (d_n, e_{j+1})$, since (d_n, e_{j+1}) does not block μ.

Let us prove that $(d_t, e_{j+1}) = (d_{\bar{t}_{p-1}}, e_{\bar{s}_{p-1}})$. If $e_{\bar{s}_{p-1}} \succ_E e_{j+1}$, then $e_{\bar{s}_{p-1}} \succ_E e_j$ or $e_{\bar{s}_{p-1}} = e_j$, then $(d_{\bar{t}_{p-1}}, e_{\bar{s}_{p-1}}) \in D \times E^j$ and, as it is a pair formed by mutually acceptable agents, $(d_{\bar{t}_{p-1}}, e_{\bar{s}_{p-1}})$ blocks μ in model $M^{(n,j)}$. This contradiction arises from having supposed that $e_{\bar{s}_{p-1}} \succ_E e_{j+1}$; therefore $e_{j+1} \succ_E e_{\bar{s}_{p-1}}$ or $e_{j+1} = e_{\bar{s}_{p-1}}$.

Let us suppose that $e_{j+1} = e_{\bar{s}_p}$, i.e., $e_{j+1} = e_{s^*+1}$. Then, $(d_t, e_{j+1}), (d_n, e_{j+1}) \in B$, which is a contradiction. Then $e_{j+1} \neq e_{\bar{s}_p}$, which implies that $j = \bar{s}_{p-1} - 1$.

We will omit the proof to the next Theorem since it is symmetrical to the previous one. ■

Theorem 33 *Let $\mu \in T_q(M^{(t^*,s^*)})$ and $(i,m) \in N$ such that $i \neq n$.*

i) *If $(d_{t^*+1}, e_m) \in \tilde{B}$, the pair (i,m) is maximal for μ if and only if $(d_{t^*+1}, e_m) \in B$ and $i = \bar{t}_1 - 1$.*

ii) *If $(d_{t^*+1}, e_m) \notin \tilde{B}$, the pair (i,m) is maximal for μ if and only if $(d_{t^*+1}, e_m) \in B$ and $i = \bar{t}_2 - 1$.*

Appendix 3

To formalize the Algorithm, we introduce the definitions and notations as follows:

Definition 34 *The pair $(t^*,\ s^*)$ is maximal right for μ if $\mu \in T_q(M^{(t^*,s^*)})$ and*

$$\mu \notin T_q(M^{(t^*,s^*+1)})$$

Definition 35 *The pair $(t^*,\ s^*)$ is maximal left for μ if $\mu \in T_q(M^{(t^*,s^*)})$ and*

$$\mu \notin T_q(M^{(t^*+1,s^*)})$$

Notice that the pair (t^*, s^*) is maximal left and right for μ if and only if, the pair (t^*, s^*) is maximal for μ.

Given $^{\mu \in T_q(M^{(t^*,s^*)})}$ for $(t^*, s^*) \in \bar{K}^C$, we can define:

$$F^D(\mu) = \{(a,b) \in N : (a,b) \text{ is maximal right for } \mu \text{ and } \bar{\mu} \in S(M^{(a,b+1)}) \text{ with } \#\bar{\mu} = q+1\}$$

$$F^I(\mu) = \{(a,b) \in N : (a,b) \text{ is maximal left for } \mu \text{ and } \bar{\mu} \in S(M^{(a+1,b)}) \text{ with } \#\bar{\mu} = q+1\}$$

and

$$F(\mu) = F^D(\mu) \cup F^I(\mu)$$

Lemma 36 *If $\mu \in T_q(M^{(t^*,s^*)})$ for $(t^*, s^*) \in \bar{K}^C$, then $F(\mu) \neq \phi$.*
Proof. *Since $\mu \in T_q(M^{(t^*,s^*)})$ for $(t^*, s^*) \in \bar{K}^C$, there exists (t', s') which is a maximal pair for μ such that $\mu \in T_q(M^{(t',s')})$.*

Let us consider the model $M^{(t'+1,s')}$(also must be done to $M^{(t',s'+1)}$). As (t', s') is a maximal pair, $\mu \notin T_q(M^{(t'+1,s')})$. We will obtain the set $S(M^{(t'+1,s')})$. Lef $\bar{\mu} \in S(M^{(t'+1,s')})$ and by Lemma A1 in Appendix 1, $\#\bar{\mu} = q$ or $\#\bar{\mu} = q+1$.

Case 1 *If $\#\bar{\mu} = q+1$. Then $(t', s') \in F^I(\mu)$, where $F(\mu) \neq \phi$.*

Case 2 *If $\#\bar{\mu} = q$, there exists (t'', s'') maximal pair for $\bar{\mu}$. Applying the above procedure (as many times as necessary), we get $F(\mu) \neq \phi$. The previous affirmation is true because D and E are finite sets and the matchins of $S(M)$ have cardinality more or equal than $q+1$.*

For each $^{\mu \in T_q(M^{(t^*,s^*)})}$, with $(t^*, s^*) \in \bar{K}^C$, we can define the set:

$$\varphi(\mu) = \bigcup_{(a,b) \in F^I(\mu)} S(M^{(a+1,b)}) \cup \bigcup_{(a,b) \in F^D(\mu)} S(M^{(a,b+1)})$$

To simplify the notation we can define $J = \{\mu \in T_q(M^{(t^*,s^*)})$ with $(t^*, s^*) \in \bar{K}^C\}$ and we compute $H_0 = \bigcup_{\mu \in J} \varphi(\mu)$.

To each $\bar{\mu}$ such that $\#\bar{\mu} = q+1$, we note with $\tilde{F}^I(\bar{\mu})$ and $\tilde{F}^D(\bar{\mu})$ the sets:

$$\tilde{F}^D(\bar{\mu}) = \{(a,b) \in N : (a,b) \text{ is maximal right for } \bar{\mu} \text{ and } \hat{\mu} \in S(M^{(a,b+1)}) \text{ with } \#\hat{\mu} = q+1\}$$

$$\tilde{F}^I(\bar{\mu}) = \{(a,b) \in N : (a,b) \text{ is maximal left for } \bar{\mu} \text{ and } \hat{\mu} \in S(M^{(a+1,b)}) \text{ with } \#\hat{\mu} = q+1\}$$

and

$$\tilde{F}(\bar{\mu}) = \tilde{F}^D(\bar{\mu}) \cup \tilde{F}^I(\bar{\mu})$$

Notice that, if $\#\mu \geq q+1$ for $\mu \in S(M)$, similarly to the previous lemma, we can prove that $\tilde{F}(\bar{\mu}) \neq \phi$.

To each $\bar{\mu} \in H_0$ we can define the sets: $\tilde{\varphi}_1(\bar{\mu}) = \underset{(a,b)\in\tilde{F}^I(\bar{\mu})}{\cup} S(M^{(a+1,b)}) \cup \underset{(a,b)\in\tilde{F}^D(\bar{\mu})}{\cup} S(M^{(a,b+1)})$ and $H_1 = \underset{\mu\in H_0}{\cup} \tilde{\varphi}(\bar{\mu})$. Sequentially to each $\hat{\mu} \in H_{i-1}$ with $i \geq 2$ we can define the sets: $\tilde{\varphi}_i(\hat{\mu})$ and $H_i = \underset{\mu\in(H_{i-1}-H_{i-2})}{\cup} \tilde{\varphi}(\hat{\mu})$. Finally we define $H = \underset{i}{\cup} H_i$. ■

The following example shows how to construct H.

Example 37 *Let $M_U^3 = (M, R_U, 3)$ with $D = \{d_1, d_2, d_3\}$, $E = \{e_1, e_2, e_3\}$, and the preferences:*

$$\begin{array}{ll} P_{d_1} = e_1, e_3, e_2 & P_{e_1} = d_2, d_3, d_1 \\ P_{d_2} = e_2, e_3, e_1 & P_{e_1} = d_1, d_3, d_2 \\ P_{d_3} = e_3, e_2, e_1 & P_{e_1} = d_1, d_2, d_3 \end{array}$$

and the following individual preferences:

$$d_1 \succ_D d_2 \succ_D d_3 \text{ and } e_1 \succ_E e_2 \succ_E e_3$$

If we compute $S(M_U^2)$, by definition we obtain $S(M_U^2) = \{\mu_1, \mu_2\}$, where

$$\mu_1 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi \\ e_1 & e_2 & \phi & e_3 \end{pmatrix} \text{ and } \mu_2 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi \\ e_2 & e_1 & \phi & e_3 \end{pmatrix}$$

and $T_2(M^{(2,2)}) = \{\mu_1, \mu_2\}$, $T_2(M^{(2,3)}) = \{\mu_1\}$, $T_2(M^{(3,2)}) = \{\mu_2\}$.

By applying $S(M_U^3)$, *by definition we find* $S(M_U^3) = T_3(M^{(3,3)}) = \{\mu_3, \mu_{4,}\mu_5\}$ *and*

$$\mu_3 = \begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \end{pmatrix}, \quad \mu_4 = \begin{pmatrix} d_1 & d_2 & d_3 \\ e_3 & e_1 & e_2 \end{pmatrix}, \quad \mu_5 = \begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_3 & e_2 \end{pmatrix}$$

and $T_3(M^{(3,3)}) = \{\mu_3, \mu_{4,}\mu_5\}$.

To find $\varphi(\mu)$ *for each* $\mu \in S(M_U^2)$, *let* $\mu_1 \in T_2(M^{(2,2)})$ *with* $(2,2) \in \bar{K}^C$, *then there exists the maximal* $(3,2)$ *such that* $\mu_1 \in S(M^{(3,2)})$. *We consider the model* $M^{(3,3)}$ *and obtain its set of stables:* $S(M^{(3,3)}) = \{\mu_3, \mu_{4,}\mu_5\}$.

As $\#\mu_3 = 3$, *we can say that* $(3,2) \in F^D(\mu_1)$. *Given* $\mu_2 \in T_1(M^{(2,2)})$; *we find the maximals for* μ_2, *which is the pair* $(2,3)$. *We consider the model* $M^{(3,3)}$ *and obtain its set of stables. As* $\#\mu_3 = 3$, *we can say that* $(2,3) \in F^I(\mu_2)$. *Therefore* $\varphi_1(\mu_1) = \varphi_1(\mu_2) = \{\mu_3, \mu_4, \mu_5\}$ *and* $H_0 = H = \{\mu_3\ ,\ \mu_4\ ,\ \mu_5\}$.

Theorem 38 *Given the matchings* $\mu_j \in T_q(M^{(t^*,s^*)})$ *for* $(t^*, s^*) \in \bar{K}^C$, *then* $H = T_{q+1}(M)$.

Proof. *Let us prove* $H \subseteq T_{q+1}(M)$ *first. If* $\mu' \in H = \underset{i}{\cup} H_i$, *then there exists* i *such that* $\mu \in H_i$.

1) *If* $\mu' \in H_0 = \underset{\mu \in J}{\cup} \varphi(\mu)$, *then there exists* $\mu \in J$ *such that* $\mu' \in \varphi(\mu)$. *By definition of* $\varphi(\mu)$ *we have that* $\#\mu' = q + 1$, *therefore* $\mu' \in T_{q+1}(M)$.

2) *If* $\mu' \in H_i$ *with* $i \neq 0$, *then there exists* $\mu \in H_{i-1}$ *such that* $\mu' \in \tilde{\varphi}(\mu)$. *By definition of* $\tilde{\varphi}(\mu)$ *we have that* $\#\mu' = q + 1$, *therefore* $\mu' \in T_{q+1}(M)$.

Let us prove now $T_{q+1}(M) \subseteq H$. *Let* $\mu \in T_{q+1}(M) = \underset{(t^*,s^*) \in K}{\dot{\cup}} T_{q+1}(M^{(t^*,s^*)})$. *Then, there exists a model* $M^{(t^*,s^*)}$ *such that* $\mu \in S(M^{(t^*,s^*)})$ *and* $\#\mu = q + 1$. *By Proposition A4 (Appendix1) we can assume* $\mu(d_{t^*}) \neq \phi$ *and* $\mu(e_{s^*}) \neq \phi$. *Then* $\mu \notin S(M^{(t^*-1,s^*)})$ *and* $\mu \notin S(M^{(t^*,s^*-1)})$, *because* μ *has assigned agents which are not considered in the models.*

Since the set of stable matchings is non-empty, then $S(M^{(t^{*-1},s^*)}) \neq \phi$ *and* $S(M^{(t^*,s^*-1)}) \neq \phi$. *Let* $\mu'^{(t^{*-1},s^*)})$ *and* $\mu''^{(t^*,s^*-1)})$, *by Lemma A1, Appendix 1,* ($\#\mu' = q+1$ *or* $\#\mu' = q$) *and* ($\#\mu'' = q+1$ *or* $\#\mu'' = q$).

The following cases may occur:

Case 1 $\mu'^{(t^*-1,\, s^*)})$ *with* $\#\mu' = q$. *Then* $(t^*-1, s^*) \in F^I(\mu')$, *because* $\mu'^{(t^*-1,\, s^*)})$, $\acute{\mu} \notin S(M^{(t^*-1)+1,\, s^*)})$, *since* $\mu'(d_{t^*}) \neq \phi$ *and* $\mu'(e_{s^*}) \neq \phi$ *(theorem of the single agents), i.e.,* (t^*-1, s^*) *it is a maximal pair left and* $\mu \in S(M^{((t^*-1)+1,\, s^*)})$, *with* $\#\mu = q+1$. *Therefore* $\mu \in \varphi(\mu')$, *then* $\mu \in \bigcup_{\mu' \in J} \varphi(\mu') = H_0 \subseteq H$.

Case 2 $\mu''^{(t^*-1,\, s^*)})$ *with* $\#\mu'' = q$. *We will omit the proof since it is equal to the previous one.*

Case 3 $\mu'^{(t^*-1,\, s^*)})$ *with* $\#\mu' = q+1$. *Let*

$$(\hat{t}, \hat{s}) = \min\left\{(t,s) \ \leq \ (t^*-1,\ s^*) \ : \ \acute{\mu} \in S(M^{(t,s)})\right\}$$

As $T_{q+1}(M^{(t^*-1,\, s^*)}) \cap T_{q+1}(M^{(\hat{t}-1,\, \hat{s})}) = \phi$ *and* $T_{q+1}(M^{(t^*-1,\, s^*)}) \cap T_{q+1}(M^{(\hat{t},\, \hat{s}-1)}) = \phi$, *then* $\mu'^{(t-1,\, s)})$ *and* $\mu'^{(t,\, s-1)})$. *As the set of stable matchings is non-empty, then* $S(M^{(\hat{t}-1,\, \hat{s})}) \neq \phi$ *and* $S(M^{(\hat{t},\, \hat{s}-1)}) \neq \phi$.

Let $\tilde{\mu} \in S(M^{(\hat{t}-1,\hat{s})})$ *and* $\hat{\mu} \in S(M^{(\hat{t},\hat{s}-1)})$, *by Lemma A1 in Appendix 1,* ($\#\tilde{\mu} = q+1$ *or* $\#\tilde{\mu} = q$) *and* ($\#\hat{\mu} = q+1$ *or* $\#\hat{\mu} = q$).

The following subcases may occur:

Case 3.1 $\tilde{\mu} \in S(M^{(\hat{t}-1,\hat{s})})$ *with* $\#\tilde{\mu} = q$, *then* $(\hat{t}-1, \hat{s}) \in F^I(\mu')$, *because* $\tilde{\mu} \in S(M^{(\hat{t}-1,\hat{s})})$, $\tilde{\mu} \notin S(M^{((\hat{t}-1)+1,\hat{s})})$ *with* $\#\mu' = q+1$, *because of that* $\tilde{\mu} \in H_0$. *We obtain all the maximal pairs of* $\tilde{\mu}$ *and between them we find* (t^*-1, s^*) *with* $\acute{\mu} \in S(M^{(t^*-1,\, s^*)})$. *Since* $\acute{\mu} \notin S(M^{(t^*,\, s^*)})$ *and* $\mu \in S(M^{(t^*,\, s^*)})$ *with* $\#\mu = q+1$, *then* $(t^*-1, s^*) \in \tilde{F}^I(\mu')$. *Therefore* $\mu \in H_i$ *for some* i. *Then* $\mu \in H$.

Case 3.2 $\hat{\mu} \in S(M^{(\hat{t}-1,\hat{s})})$ *with* $\#\hat{\mu} = q$. *We will omit the proof since it is equal to the previous one.*

Case 3.3 $\tilde{\mu} \in S(M^{(\hat{t}-1,\hat{s})})$ *with* $\#\tilde{\mu} = q+1$. *We apply the same reasoning of the beginning of the case 3, i.e., are* $\tilde{\mu} \in S(M^{(\hat{t}-1,\hat{s})})$ *and* $(\breve{t}, \breve{s}) = \min\left\{(t,s) \leq (\hat{t}-1, \hat{s}) : \mu'^{(t,s)})\right\}$ *with* $\tilde{\mu} \notin S(M^{(\breve{t}-1, \breve{s})})$ *and* $\tilde{\mu} \notin S(M^{(\breve{t}, \breve{s}-1)})$. *Since the set of stable matchings is non-empty, then* $S(M^{(\breve{t}-1, \breve{s})}) \neq \phi$ *and* $S(M^{(\breve{t}, \breve{s}-1)}) \neq \phi$.

Let $\tilde{\mu}_1 \in S(M^{(\breve{t}-1, \breve{s})})$ *and* $\hat{\mu}_1 \in S(M^{(\breve{t},\breve{s}-1)})$, *by Lemma A1 in Appendix 1, (*$\#\tilde{\mu}_1 = q+1$ *or* $\#\tilde{\mu}_1 = q$*) and (*$\#\hat{\mu}_1 = q+1$*or* $\#\hat{\mu}_1 = q$*).*

The following subcases may occur. It can be observed according to the result, that the study of case 3 is repeated, with a similar process according to the changes of the reduced model we choose and this situation occurs until, in the model obtained we find that the cardinal matching stable is q, and the proof will be similar to case 1.

Case 3.4 $\hat{\mu} \in S(M^{(\hat{t}-1,\hat{s})})$ *with* $\#\hat{\mu} = q+1$. *We will omit the proof since it is equal to the previous one.*

Case 4 $\acute{\mu} \in S(M^{(t^*-1, s^*)})$, *with* $\#\mu'' = q+1$. *The proof is similar to case 3.*

■