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Cyclic Games

by

Luis Guillermo Quintas (*)

Abstract:

In this paper we consider Nash equilibria in Q-Cyclic n-person games. This family of games was introduced by (Marchi and Quintas (1983)). It is a subfamily of the Polymatrix Games (Yanovskaya (1968)). We analyze properties of Q-Cyclic games. We review the conditions of uniqueness and we present the construction of families of games with unique Nash equilibrium. We also mention the relation of these games with a dynamic model for analyzing corruption evolution (Accinelli, Martins, Oviedo, Pinto and Quintas (2017)).

Key Words: N-person games – Q-Cyclic games – Uniqueness.

(*) IMASL Instituto de Matematicas Aplicadas San Luis - CONICET, Departamento de Matemáticas, Facultad de Ciencias Físico - Matemáticas y Naturales, Universidad Nacional de San Luis, Ejército de Los Andes 950, 5700 San Luis, Argentina. E-Mail: lu6quintas@gmail.com

1.- Introduction

The concept of equilibrium studied by Nash (1950) is considered a landmark in non-cooperative game theory. Mixed strategies are probability distributions over the pure strategies of each player, and the utilities functions are the corresponding expectations of the players. The set of Nash equilibria is non empty for any finite game if mixed strategies are allowed (Nash (1950)). This theorem of existence of Nash equilibrium was obtained by using Kakutani fixed point theorem (Kakutani (1941)). Since this fundamental result which deserved John Nash to obtain the Nobel Prize in 1994, several issues have been studied on computation of equilibrium, on the analysis of structure of the Nash equilibrium set, studies of multiplicity and uniqueness of equilibrium, etc.

Zero sum games are a subfamily of the n-person games where the sum of all players' utilities is zero[1]. Thus each player´s gain or loss of utility is balanced by the losses or gains of the utility of the other players. These games have been widely studied, and for two-player finite zero sum games, Nash equilibrium, minimax, and maximin concepts give all the same solution (Von Neumann and Morgenstern (1947)).

Interchangeability of equilibrium strategies (Nash (1951)) states that the players can choose strategies of any equilibrium and they will remain in equilibrium. This property holds for two players zero sum games. It also holds for two players strictly competitive games. In these games when a player improves the utilities the other player has a reduction of the utility level.

Bimatrix games stand for general two players games. Nash equilibrium in bimatrix games can be computed by using the Lemke–Howson algorithm (1964). The Nash equilibrium set for these games is a finite union of products of convex polytopes. The structure of the Nash equilibrium set in general n-person games is more complex.

Q-cyclic n-person games (Marchi and Quintas (1983)) are n-person games were the players play "pairwise" conforming a cycle of players. We will analyze some properties of these games, about the Nash equilibrium set (which

[1] Any game where the sum is a constant c, can be transformed into a zero-sum game.

results similar to the Bimatrix games) and the cases of uniqueness of Nash equilibrium.

We also consider some studies presented by Quintas (1989) on Polymatrix Games (Yanovskaya (1968)) and the studies of constructions of games with equilibrium done by Alaniz and Quintas (2010) and Alaniz, Quintas and Sevilla (2014).

Finally, we mention the relation of Cyclic games with a dynamic model for analyzing corruption evolution (Accinelli, Martins, Oviedo, Pinto and Quintas (2017)).

2.- Cyclic Games and Polymatrix Games

Here we give some definitions, notations, and we review some results which will be used in this paper.

Definition 2.1 (Marchi - Quintas (1983)). Let $\Gamma = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n, A_1, A_2, \ldots, A_n\}$, be a finite n-person Q-cyclic game in normal form, where Σ_i is the set of pure strategies for player i. Let A_i be the utility function of player i, with i=1,...n. The definition of the function A_i is given by: $A_i(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_n) = B_i(\sigma_i, \sigma_{q(i)})$ with $\sigma_i \in \Sigma_i$ where the function q is that: $q(i) \neq i$ and $|q^{-1}(i)| = 1$ were $|.|$ stands for the cardinality of respective set .

In this work we consider games where: q(i) = i+1 mod. n , we take j = q(i) and each player having m strategies, thus $|\Sigma_i| = m$, for i=1,.....,n. This type of games was denominated *Purely Q-Cyclic Games* in the article by Marchi and Quintas (1983), and we will refer in this paper as *Cyclic Games*.

Definition 2.2: A mixed strategy for players i, is a probability distribution over the set of pure strategies $\Sigma_i = \{\sigma_1^i, \sigma_2^i, \ldots, \sigma_m^i\}$. That is a vector : $x_i = (x_i(\sigma_1^i), x_i(\sigma_2^i), \ldots, x_i(\sigma_m^i)) = (x_1^i, x_2^i, \ldots x_m^i)$ where x_t^i is the probability of player i uses his strategies $\sigma_t^i \in \Sigma_i$ with t = 1,.....,m.

Definition 2.3: Let $\tilde{\Sigma}_i$ be the set of mixed strategies for the player i .

$$\tilde{\Sigma}_i = \left\{ x_i : \sum_{t=1}^{t=m} x_t^i = 1 \quad con \quad x_t^i \geq 0, t = 1,2,.....,m \right\}$$

$x = (x_1, x_2, ...x_n) \in \prod_{i=1}^{i=n} \tilde{\Sigma}_i$ is a n-tuple of mixed strategies for the n players and we denote $\left(x^*_{N-\{i\}}, x_i\right) = \left(x^*_1, ..., x^*_{i-1}, x_i, x^*_{i+1}, ..., x^+_n\right)$

Definition 2.4 (Marchi - Quintas (1983)). The expected utility function E_i for each player i, in the Cyclic game is defined as follows:
$E_i(x) = F_i(x_i, x_{q(i)})$ where F_i is the expected utility function of B_i.

Thus, we have $E_i(x) = F_i(x_i, x_j) = \sum_{l=1}^{l=m} \sum_{t=1}^{t=m} a_{tl}^{ij} x_l^j x_t^i$ with i=1, 2,......,n

We will indistinctly denote it by: $F_i(r, x_j) = F_i(e_r, x_j)$, where e_r is a m-tuple with one in the place r and zero in the other places.

Definition 2.5: (Nash (1950)). A n-uple $x^* = (x_1^*, x_2^*, ...x_n^*) \in \prod_{t=1}^{t=n} \tilde{\Sigma}_t$ is a Nash equilibrium if and only if $E_i(\overset{*}{x}) \geq E_i(\overset{*}{x}_{N-\{i\}}, x_i)$ for each $x_i \in \tilde{\Sigma}_i$; and for each i=1,. ,n.

We will use the following characterization on Nash Equilibrium:

Definition 2.6: The set of all the pure strategies that are best reply against $x = (x_1, x_2, ...x_n) \in \prod_{t=1}^{t=n} \tilde{\Sigma}_t$, is defined as follows:

$$J_i(x_{q(i)}) = \left\{ \sigma_r^i \in \Sigma_i : F_i(\sigma_r^i, x_j) \geq F_i(\sigma_t^i, x_{q(i)}) \text{ for each } \sigma_t^i \in \Sigma_i \text{ with } t = 1,2,....,m \right\}$$

We will use the following characterization of equilibrium points:

Theorem 1: $x=(x_1,x_2,......x_n)\in\prod_{i=1}^{i=n}\tilde{\Sigma}_i$ is a Nash equilibrium, if and only if,

$S(x_i)\subseteq J_i(x_{q(i)})$, for each i = 1,2,....., n.

$S(x_i)=\{\ \sigma_s^i\in\Sigma_j : x_s^i>0$ with $s=1,2,...,m\ \}$ is the support of the mixed strategy x_i.

Definition 2.7: $x=(x_1,x_2,......x_n)\in\prod_{i=1}^{i=n}\tilde{\Sigma}_i$ is completely mixed if $S(x_j)=\Sigma_j$ for each i = 1,2,....., n. In this case we say that each player has all the strategies active.

Definition 2.8: Let $x^*=(x_1^*,x_2^*,......x_n^*)\in\prod_{t=1}^{t=n}\tilde{\Sigma}_t$ be a Nash Equilibrium of $\tilde{\Gamma}$. v_1, v_2,......, v_n ., are expected values associated to $x^*=(x_1^*,x_2^*,......x_n^*)\in\prod_{t=1}^{t=n}\tilde{\Sigma}_t$ where $v_i = E_i(x^*)$ for i=1,2,....,n

Definition 2.9 A Polymatrix game (Yanovskaya (1968)) is a n-person non cooperative normal form game $\Gamma=\{\Sigma_1,\Sigma_2,.....,\Sigma_n,A_1,A_2,.......,A_n\}$. If player i chooses strategy $\sigma_i\in\Sigma_i$, and player j chooses strategy $\sigma_j\in\sum_j$ it is possible to assign a partial payoff $a^{ij}(\sigma_i,\sigma_j)$ such that for any choice of pure strategies $(\sigma_1,...,\sigma_n)$ by the n players, the payoff to player i is given by:

$$A_i(\sigma_1,...,\sigma_n)=\sum_{j\neq i}a^{ij}(\sigma_i,\sigma_j)$$

3.- Uniqueness of Nash equilibrium in Cyclic games

The theorem of existence of Nash equilibrium proved by Nash (1950), opened several research lines focusing on the important problem to decide which equilibrium is taken as a solution of the game. Thus, it appeared refinements of the Nash equilibrium in order to reduce the multiplicity of equilibria and studies of uniqueness.

If the players cannot communicate, each one might choose an equilibrium strategy and the resulting play might not be an equilibrium[2]. Even if they can communicate, it still remains a serious problem because the utilities can be quite different from one equilibrium point to another. This problem does not arise if the equilibrium is unique

Many studies have been done on uniqueness of Nash equilibrium points. On one hand it was studied sufficient conditions to guarantee uniqueness[3]. it has been also studied under what conditions it is possible to construct games with predetermined unique equilibrium predetermined. Constructions of games with prefixed unique equilibrium have been done for bimatrix games by Raghavan (1970). He proved that, if the equilibrium points of a game are completely mixed then the matrix of each player is square and the equilibrium is unique. Millham (1972) proved that a necessary and sufficient condition for the existence of a game with unique prefixed equilibrium points is that the equilibrium be completely mixed. Kreps (1974) gave uniqueness conditions when the equilibrium point is not completely mixed. Heuer (1979), extended and complemented these results and obtained the uniqueness of the equilibrium point within the class of mixed strategies whose non zero components are the same for all the players. Quintas (1988 a) b)) extended this result constructing a wide family of bimatrix games with unique equilibrium point. Marchi and Quintas (1987) also studied games with prefixed values and unique equilibrium points. Quintas, Marchi, Giunta and Alaniz (1991) extended this construction for other families of games with unique equilibrium points.

We will present a general form of n-person q-cyclic games with prefixed equilibrium points on the mixed extension. We will construct the payoff matrices A_i with i=1,2,...,n for each player i, and we will study under what conditions on the expected utility function E_i there is a unique equilibrium.

Let us consider an arbitrary point, $x=(x_1,x_2,......x_n)\in\prod_{i=1}^{i=n}\tilde{\Sigma}_i$

It is $(x_1,x_2,.........,x_n)=\left((x_1^1,x_2^1,........,\ x_m^1),(x_1^2,x_2^2,........,\ x_m^2),.........\ ....(x_1^n,x_2^n,........,\ x_m^n)\right)$ with

[2] This problem does not appear in two person zero sum games because for this type of games the equilibrium could not be unique, but the equilibrium strategies are interchangeable and the equilibrium payoff is unique

[3] There is also a vast bibliography on refinements of the Nash Equilibrium. Some refinements reduce the multiplicity of equilibrium.

$\sum_{t=1}^{t=m} x_t^i = 1$ for each i=1,2,....,n , $x_t^i > 0$ for each t=1,2,....,m and i=1,2,.....,n

Thus, $|S(x_i)| = m$ for i=1,2,....n.

We choose arbitrary non zero values v_i with i=1,.......,n. They will be the expected payoffs of the game

The construction extends that presented for bimatrix games by Quintas (1988 a)). It consists in giving conditions in order that the region of the simplex $\tilde{\Sigma}_j$ limited for a: predetermined vertex, the above prefixed point x and some points chosen on the faces of the simplex, have a unique maximizing hyperplane.

Thus, we take into consideration the prefixed point $x_j = \left(x_1^j, x_2^j, ...x_m^j\right) \in \Sigma_j$ and the simplex´s vertexes e_s , having one in the place s and zero in the other places.

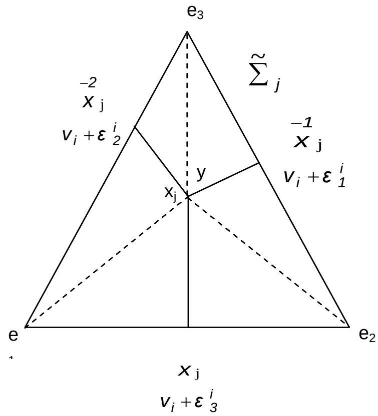
We choose s points having the following form:

$$\overset{-s}{x_j} = (\overset{-j}{x_1}, \overset{-j}{x_2}, ..., \overset{-j}{x_{s-1}}, 0, \overset{-j}{x_{s+1}}, ..., \overset{-j}{x_m}) \quad \text{with } s=1,2,......m$$

The point $\overset{-s}{x_j}$ is on a face of simplex $\tilde{\Sigma}_j$, and we obtain it by extending the segment between e_s and x_j, until it reaches the opposite face to the corresponding vertex.

This is made in order to obtain a polyhedral partition of the simplex $\tilde{\Sigma}_j$, having as extreme points: the m vertices of simplex $\tilde{\Sigma}_j$, the m points $\overset{-s}{x_j}$ and the prefixed point x_j . (Marchi and Quintas (1987) studied characterizations of these points on some n-person games).

The geometric idea laying behind the construction of the payoff matrixes consists in analyzing which is the “maximizing hyperplane” in each subset of the simplex partition. We want to have a unique “maximizing hyperplane” in each region (see figure when $|\Sigma_j| = 3$)



In order to obtain $\overset{-s}{x_j}$ we use the following equation:

$$e_s + \lambda^s\left(x_j - e_s\right) = \overset{-s}{x_j}$$

As the s-th component of $\overset{-s}{x_j}$ is cero, then we have: $1+\lambda_S\left(x^j_S - 1\right) = 0$

Thus $\lambda^s = \dfrac{1}{1-x^j_s} > 0$, and it follows that: $\overset{-s}{x_t} = \begin{cases} \dfrac{x^j_t}{1-x^j_t} > 0 & \text{for each } t \neq s \\ 0 & \text{for each } t = s \end{cases}$

Thus: $\overset{-s}{x}_j = \left(\dfrac{x^j_1}{1-x^j_s}, \dfrac{x^j_2}{1-x^j_s}, \ldots\ldots, \dfrac{x^j_{s-1}}{1-x^j_s}, 0, \dfrac{x^j_{s+1}}{1-x^j_s}, \ldots\ldots, \dfrac{x^j_m}{1-x^j_s}\right)$

As $1-x^j_s = \sum_{t\neq s} x^j_t$ we have:

$$\overset{-s}{x_j} = \frac{1}{\sum_{t\neq s} x^j_t}\left(x^j_1, x^j_2, \ldots\ldots, x^j_{s-1}, 0, x^j_{s+1}, \ldots\ldots, x^j_m\right)$$

For each vertex $e_r \in \tilde{\Sigma}_i$ and for each xj $\in \tilde{\Sigma}_j$, by definition 6 we obtain:

$$F_i\left(r, x_j\right) = \sum_{l=1}^{l=m} a^{ij}_{rl}\, x^j_l = A^r_i\, x^t_j,$$

where A^r_i is the r-th row of the matrix A_i of player i and x^t_j is the transposed of x_j.

On the hyperplane $F_i(r, x_j)$ we require the following properties:

- It should take the value v_i on $x_j = (x_1^j, x_2^j, ...x_m^j) \in \Sigma_j$, . That is:

$$F_i(r, x_j) = \sum_{l=1}^{l=m} a_{rl}^{ij}\ x_l^j = A_i^r\ x_j^t = v_i$$

- In each point $\overset{-s}{x}_j$ with s= 1, 2,.....,n, it should take the value $v_i + \varepsilon_s^i$, with $\varepsilon_s^i > 0$. Namely:

$$F_1\left(r, \overset{-s}{x}_j\right) = \sum_{\substack{l=1 \\ l \neq s}}^{l=m} a_{rl}^{ij}\ x_l^j = v_i + \varepsilon_r^i$$

Then $\dfrac{1}{1 - x_s^j} \sum_{\substack{l=1 \\ l \neq s}}^{l=m} a_{rl}^{ij} = v_i + \varepsilon_s^i$

Now we introduce a bijective function in order to complete the definition of player i payoff matrix.

$$f_{ji} : \Sigma_j \to \Sigma_i\ ,\ \ f_{ji}(s) = r \text{ such that } F_i(r, x_j) = v_i$$

(where r is the index of the corresponding maximizing hyperplane)

This implies that:

$$F_i(r, s) = a_{rs}^{ij} > a_{ts}^{ij} = F_i(t, s) \quad \text{for each } t \in \tilde{\Sigma}_i$$

Thus we have: $F_i(r, s) = a_{rs}^{ij}$ and by Theorem 1, we obtain: $J_i(e_s) = \{f_{ji}(s)\}$

In this way in each vertex of $\tilde{\Sigma}_j$ there is a unique maximizing hyperplane and f_{ji} distributes the different hyperplanes on the different vertices.

We also prescribe that each t, such as $f_{ji}(t) \neq r$, the $f_{ji}(t)$-hyperplane, "passes underneath" the $f_{ji}(s)$-hyiperplane at each point $\overset{-s}{x}_j$. Moreover we ask it takes the values $v_i + \varepsilon_s^i$.

Thus, for each r $\neq f_{ji}(t)$.

$$\sum_{l=1}^{l=m} a_{rl}^{ij} \overset{-s}{x_l} = \frac{1}{1-x_s^j} \sum_{\substack{l=1 \\ l\neq s}}^{l=m} a_{rl}^{ij} x_l^j = v_i + \varepsilon_s^i$$

$$>$$

$$\sum_{l=1}^{l=m} a_{f_{ji}(s)l}^{ij} \overset{-s}{x_l} = \frac{1}{1-x_s^j} \sum_{\substack{l=1 \\ l\neq s}}^{l=m} a_{f_{ji}(s)l}^{ij} x_l^j$$

Thus for r=1,2,...,m in the point $x_j = \left(x_1^j, x_2^j, \ldots\ldots, x_m^j\right)$ we prescribe that all the hyperplanes take the same value v_i. This is, for r =1,2,...,n,

$$\sum_{l=1}^{l=m} a_{rl}^{ij} x_l^j = v_i$$

And for each t such that $f_{ji}(t) \neq r$

$$\frac{1}{1-x_t^j}\left[\left(\sum_{l=1}^{l=3} a_{rs}^{ij}\; x_l^j\right) - a_{rt}^{ij} x_t^j\right] = v_i + \varepsilon_t^i$$

Then, we have the following system:

$$\begin{cases} \displaystyle\sum_{l=1}^{l=m} a_{rl}^{ij} x_l^j = v_i \\ y \quad \forall t : f_{ji}(t) \neq q \\ \displaystyle\frac{1}{1-x_t^j}\left[\left(\sum_{l=1}^{l=m} a_{rl}^{ij} x_l^j\right) - a_{rt}^{ij} x_t^j\right] = v_i + \varepsilon_t^i \end{cases}$$

Solving it, we have:

$$a_{rs}^{ij} = \begin{cases} v_i + \dfrac{1}{x_s^j} \displaystyle\sum_{f_{ji}(t)\neq s} \varepsilon_t^i \left(1 - x_t^j\right) & \text{para } f_{ji}(s) = r \\ v_i - \dfrac{\left(1 - x_s^j\right)\varepsilon_s^i}{x_s^j} & \text{para } f_{ji}(s) \neq r \end{cases}$$

Remark 1:

- As $0 < x_s^j < 1$ the coefficients are well defined.
- As $f_{ji}(s) = r$, then $a_{rs}^{ij} > v_i$.

- As $f_{ji}(s) \neq r$; then $a^{ij}_{rs} < v_i$.

These two inequalities agree with the geometric idea leading the whole construction.

- The payoff matrix A_i is non singular, its determinant is:

$$det(A_i) = \left(\sum_{s=1}^{s=m} (1 - x^i_s)\ \varepsilon^i_s \right)^{n-1} \frac{v_i}{x_1.x_2.x_3......x_n}$$

And as ε^i_s are arbitrary positives numbers, $\sum_{s=1}^{s=m} (1 - x^i_s)\ \varepsilon^i_s > 0$. Furthermore, v_i is not null, then $det(A_i) \neq 0$.

4.- Existence and Uniqueness of Nash Equilibrium

In this section we present the construction presented by Alaniz S. and Quintas L. G. (2010). We check here that the point $x = (x_1, x_2, ...x_n) \in \prod_{i=1}^{i=n} \tilde{\Sigma}_i$, with $x^j_s > 0$ for s= 1,..., m , is a Nash Equilibrium. It is so, because it fulfills the inclusions given, in Theorem 1:

$$S(x_i) \subseteq J_i(x_{q(i)}), \text{ for each } i = 1,2,....., n$$

As all the hyperplanes of player i take the same value v_i in the point x_j, and taking into account that $\sum_{l=1}^{l=m} a^{ij}_{rl} x^j_l = v_i$; then $J_i(x_j) = \{1, 2, ..., n\} = S(x_i)$.

Thus, the construction presented in the previous section guarantees the existence of a completely mixed Nash Equilibrium $(x_1, x_2, ..., x_n)$ for a Q-cyclic game Γ with payoff matrix A_i for player i.

Theorem 2: Given $(x_1, x_2, ..., x_n) = ((x^1_1, x^1_2, ..., x^1_m), (x^2_1, x^2_2, ..., x^2_m), ... (x^n_1, x^n_2, ..., x^n_m))$ with $\sum_{t=1}^{t=m} x^i_t = 1$ for each i=1,2,....,n with $x^i_t > 0$ for each t=1,2,....,m and for each i=1,2,.....,n; and values nonzero $v_1, v_2,...,v_n$. given the functions $f_{ji} : \Sigma_j \to \Sigma_i$, with j= i+1 mod. n with positives numbers ε_s^i i=1, 2, ,....,n: There exists a Q-cyclic game Γ having $(x_1, x_2,......., x_n)$ as a completely mixed Nash Equilibrium with payoff values $v_1, v_2,...,v_n$

We will prove that the family of games we constructed in the previous section has $(x_1, x_2, ...x_n) \in \prod_{t=1}^{t=n} \tilde{\Sigma}_t$ as unique Nash Equilibrium.

Given the payoff matrices A_1, A_2,.......,A_n for the corresponding players, we choose functions f_{ji} and Φ fulfilling that for each $(r', r'') \in \Sigma_j x \Sigma_j$:

$$\text{If} \quad \Phi \circ f_{ji}(r') = r'' \quad then \quad \Phi \circ f_{ji}(r'') \neq r' \quad (1)$$

with $\Phi : \Sigma_i \to \Sigma_j$ resulting of the compositions of functions of the type f_{ji}.

Remark 2:

- These conditions are similar to that given for Bimatrix games by Quintas (1988 b),
- In a 3-person game, player 1 plays with player 2, and condition (1) takes the form:

  For each $(j', j'') \in \Sigma_2 x \Sigma_2$:

  $$\text{If} \quad \Phi \circ f_{21}(j') = j'' \quad then \quad \Phi \circ f_{21}(j'') \neq j'$$

  In this case $\Phi = f_{32} \circ f_{13}$.

- The functions $f_{ji}(r) = r$ and $\Phi(r) = r - 1$ mod.m fulfill condition (1).

Moreover, if we choose positive number ε_s^i we have that:

$$\sum_{s=1}^{s=m} (1 - x_s^j) \varepsilon_s^i > 0 \qquad (2)$$

This implies that the payoff matrix of each player i is non singular.

We will use the following notation:

- $(x_j)^t$ is the transposed vector of x_j.
- $V_i = (v_i \; v_i .... v_i)^t$ is a (mx1) matrix with v in all entries.
- A_i^j is the j-th row of player i payoff matrix

Theorem 3:

Given a Q-cyclic game Γ constructed as in Theorem 2, having $(x_1, x_2,......., x_n)$ as a completely mixed Nash Equilibrium, with non zero values $v_1, v_2,...,v_n$, and given functions f_{ji} fulfilling condition (1) and given positive numbers ε_s^i with i=1, 2, ,....,n fulfilling condition (2) , then, $(x_1, x_2,......., x_n)$ is the unique Nash Equilibrium of the game Γ.

Proof:

In order to prove the uniqueness, we assume that there exists another Nash equilibrium $(y_1,y_2,.......,y_n)$ with values u_i , thus we have the expected utilities $E_i(y_1,y_2,.......,y_n) = u_i$ for i=1,2,...n.

We will consider the following cases:

- Case 1: $(y_1,y_2,.......,y_n)$ is completely mixed

As it is a Nash equilibrium it fulfills the systems:

$$A_i(x_j)^t = V_i \qquad\qquad A_i(y_j)^t = U_i$$

Multiplying each equation of system a) by u_i , and multiplying each equation of system b) v_i and subtracting one system from the other we obtain:

$$A_i(u_i x_j - v_i y_j)^t = 0$$ being 0 the (mx1)-order null matrix.

The matrices A_i are non singular, because $v_i \neq 0$ and ε_p^i fulfills condition (2). Thus the lineal homogeneous systems $A_i(u_i x_j - v_i y_j)^t = 0$ has unique solution, namely the trivial. $u_i x_j - v_i y_j = 0$, It implies that, $u_i x_s^j = v_i y_s^j$ with s=1,2,.......m.

Summing up over s we obtain: $\sum_{s=1}^{s=m} u_1 x_s^j = \sum_{s=1}^{s=m} v_1 y_s^j$

As x_j and y_j are probability vectors, then $u_i = v_i$, therefore: $A_i(x_j)^t = A_i(y_j)^t$ thus $A_i(x_j - y_j)^t = 0$ and as A_i is non singular then the system has unique solution $x_j - y_j = 0$, and thus $x_j = y_j$ for each i=1,2,.....,n

- Case 2: The point $(y_1, y_2, \ldots, y_n)$ fulfills that:
$S(y_i) = S(x_i)$ with i $\in$ {1,2,.......,n}, excepting for some j $\in$ {1,2,.......,n}, such that

$S(y_j) \subseteq S(x_j)$ with $j \neq i$

We assume that $y_j = (y_1^j, y_2^j, \ldots, y_k^j, \ldots, y_{m-1}^i, 0)$ and $f_{ji}(k) = k$

Let $k \in S(x_j) - S(y_j)$,

$$A_i^{f_{32}(k)}(y_j)^t = v_i \sum_{s \in S(y_j)} y_s^j - \sum_{s \in S(y_j)} \frac{\varepsilon_s^i (1 - x_s^i)}{x_s^i} y_s^j$$

Let $r \in S(y_i)$

$$A_i^{f_{ji}(r)}(y_j)^t = v_i \sum_{s \in S(y_j)} y_s^j - \sum_{\substack{s \in S(y_j) \\ s \neq r}} \frac{\varepsilon_s^i (1 - x_s^i)}{x_s^i} y_s^j + \frac{y_r^j}{x_r^j} \sum_{\substack{s=1 \\ s \neq r}}^{s=m} \varepsilon_s^i (1 - x_s^j)$$

Subtracting $A_i^{f_{ji}(r)}(y_j)^t$ y $A_i^{f_{ji}(k)}(y_j)^t$ we obtain:

$$v_i \sum_{s \in S(y_j)} y_s^j - \sum_{\substack{s \in S(y_j) \\ s \neq r}} \frac{\varepsilon_s^i (1 - x_s^i)}{x_s^i} y_s^j + \frac{y_r^j}{x_r^j} \sum_{\substack{s=1 \\ s \neq r}}^{s=m} \varepsilon_s^i (1 - x_s^j) - \left(v_i \sum_{s \in S(y_j)} y_s^j - \sum_{s \in S(y_j)} \frac{\varepsilon_s^i (1 - x_s^i)}{x_s^i} y_s^j \right),$$

That is:

$$A_i^{f_{ji}(r)}(y_j)^t - A_i^{f_{ji}(k)}(y_j)^t = \frac{y_r^j}{x_r^j} \sum_{\substack{s=1 \\ s \neq r}}^{s=m} \varepsilon_s^i (1 - x_s^j) + \frac{\varepsilon_r^i (1 - x_r^j)}{x_r^j} y_r^j = \frac{y_r^j}{x_r^j} \sum_{s=1}^{s=m} \varepsilon_s^i (1 - x_s^j)$$

We have: $\sum_{s=1}^{s=m} \varepsilon_s^i (1 - x_s^j) > 0$ y $r \in S(y_j)$

$A_i^{f_{ji}(r)}(y_j)^t - A_i^{f_{ji}(k)}(y_j)^t > 0$, implies that $f_{ji}(k) \notin J_i(y_j)$.

Thus $(y_1, y_2, \ldots, y_n)$ is an Equilibrium point, because : $S(y_i) \subseteq J_i(y_{-j})$ (Theorem 1) , then $f_{ji}(k) \notin S(y_i)$, and in consequence $y_{f_{ji}(k)}^i = 0$., but that is impossible because by hypothesis, we had that $S(y_i) = S(x_i)$, and thus $(x_1,$

$x_2,\ldots\ldots, x_n)$ is a point completely mixed Equilibrium, which implies that $|S(y_i)| = |S(x_i)| = m$.

In consequence, $S(y_j) = S(x_j)$ for all j=1,2,...., n and then by case 1, $x_j = y_j$ therefore $(x_1, x_2,\ldots\ldots, x_n)$ is unique equilibrium.

- Case 3: The point $(y_1, y_2,\ldots\ldots, y_n)$ fulfills that:

$$S(x_i) \subseteq S(y_i) \qquad for\ \ all \quad i = 1, 2, \ldots, n$$

Let $k \in S(x_n) - S(y_n)$. In this case, we suppose k =m $f_{n(n-1)}(k) = k$,

In case 2, we obtained $y^{n-1}_{f_{n(n-1)}(k)} = 0$, thus $y^{n-1}_m = 0$.

Let $j \in S(x_{n-1}) - S(y_{n-1})$, by hypothesis we had $S(y_{n-1}) \subseteq S(x_{n-1})$, in this case j = m satisfies that condition.

Let $f_{(n-1)(n-2)}(j) = j + 1$ mod.m,

$$A^{f_{(n-1)(n-2)}(j)}_{n-2}(y_{n-1})^t = v_{n-2} \sum_{s \in S(y_{n-1}')} y^{n-1}_s - \sum_{s \in S(y_{n-1})} \frac{\varepsilon^{n-2}_s (1 - x^{n-1})}{y_s} y^{n-1}_s$$

Let $r \in S(y')$,

$$A^{f_{(n-1)(n-2)}(r)}_{n-2}(y_{n-1})^t = v_{n-2} \sum_{s \in S(y_{n-1})} y^{n-1}_s - \sum_{s \in S(y_{n-1})} \frac{\varepsilon^{n-2}_s (1 - x^{n-1}_s)}{x^{n-1}_s} y^{n-1}_s + \frac{y^{n-1}_r}{x^{n-1}_r} \sum_{\substack{s=1 \\ s \neq r}}^{s=m} \varepsilon^{n-2}_s (1 - x^{n-1}_s)$$

By subtracting $A^{f_{(n-1)(n-2)}(r)}_{n-2}(y_{n-1})^t$ from $A^{f_{(n-1)(n-2)}(j)}_{n-2}(y_{n-1})^t$ we obtain, as in the previous case that: $A^{f_{(n-1)(n-2)}(r)}_{n-2}(y_{n-1})^t > A^{f_{(n-1)(n-2)}(j)}_{n-2}(y_{n-1})^t$, then $f_{(n-1)(n-2)}(j) \notin J_{n-2}(y_{n-1})$.

As we have that $(y_1, y_2,\ldots\ldots, y_n)$ is an equilibrium $S(y_{n-2}) \subseteq J_{n-2}(y_{n-1})$ (Theorem 1), then $f_{(n-1)(n-2)}(j) \notin S(y_{n-2})$, and in consequence , $y^{n-2}_{f_{(n-1)(n-2)}(j)} = 0$, that is $y^{n-2}_1 = 0$.

Let $i \in S(x_1) - S(y_1)$, and we consider $f_{1n}(i) = i$

Working as in the previous cases we obtain that $f_{1n}(i) \notin J_n(y_1)$.

Again as $(y_1, y_2, \ldots\ldots, y_n)$ is an equilibrium $S(y_n) \subseteq J_n(y_1)$ (Theorem 1) , then $f_{1n}(i) \notin S(y_n)$, and in consequence: $y^n_{f_{1n}(i)} = 0$, that is is $y^n_1 = 0$.

Now by choosing $k \in S(x_n) - S(y_n)$, such that: $L(k) = f_{1n}(f_{21}(\varphi(f_{n(n-1)}(k)))) \in S(y_n)$,

The existence of such k, is guarantied by (1).

For k= m, $L(m) = 1$

$$A_n^{L(k)}\left(y_1{'}\right)^t = v_n \sum_{s \in S(y_1)} y_s^1 - \sum_{s \in S(y_1)} \frac{\varepsilon_s^n \left(1 - x_s^1\right)}{x_s^1} y_s^1$$

and for each r such that $h(r) = f_{21}(\varphi(f_{n(n-1)}(r))) \in S(y_1)$, in consequence, $\Phi(r) \neq 1$

$$A_n^{L(r)}(y_1)^t = v_n \sum_{s \in S(y_1)} y_s^1 - \sum_{\substack{s \in S(y_1) \\ s \neq h(r)}} \frac{\varepsilon_s^n \left(1 - x_s^1\right)}{x_s^1} y_s^1 + \frac{y^1_{h(r)}}{x^1_{h(r)}} \sum_{\substack{s=1 \\ s \neq h(r)}}^{s=m} \varepsilon_s^n \left(1 - x_s^1\right) \text{ i}$$

Subtracting $A_n^{L(r)}(y_1)^t$ from $A_n^{L(k)}(y_1)^t$, we obtain:

$$A_n^{L(r)}(y_1)^t - A_n^{L(k)}(y_1)^t = \frac{y^1_{h(r)}}{x^1_{h(r)}} \sum_{s=1}^{s=m} \varepsilon_s^n \left(1 - x_s^1\right)$$

This is positive because $h(r) \in S(y_1)$ and $\sum_{s=1}^{s=m} \varepsilon_s^n \left(1 - x_s^1\right) > 0$

Thus , $A_n^{L(r)}(y_1)^t > A_n^{L(k)}(y_1)^t$ (3)

However, k is such that $L(k) \in S(y_1)$, and as $(y_1, y_2, \ldots\ldots, y_n)$ is an Equilibrium, then $S(y_n) \subseteq J_n(y_1)$ (Theorem 1), we have

$$A_n^{L(r)}(y_1)^t \leq A_n^{L(k)}(y_1)^t . \quad (4)$$

But the inequalities (3) and (4) are incompatible, unless $y^1_{h(r)} = 0$.

If this occurs the vector y_1 is the null, and this is a contradiction, then it doesn't exist other equilibria of the game Γ. It completes the proof ■

Alaniz, Quintas and Sevilla (2014) presented another family of Q-Cyclic games with unique Nash equilibrium. The corresponding payoff matrices are the following:

$$a_{rs}^{ij} = \begin{cases} v_i + \varepsilon_i \dfrac{(m-1)}{m}\left(2(m-1) + \displaystyle\sum_{\substack{k,s=1\\ k\neq s\\ k\neq r}}^{m} \dfrac{x_k^j}{x_s^j}\right) & \text{para} \quad f_{ji}(s) = r \\ v_i - \varepsilon_i \dfrac{(m-1)}{m}\left(-(m-2) + \dfrac{2x_r^j}{x_r^j} + \displaystyle\sum_{\substack{s\neq r\\ s\neq p}}^{m} \dfrac{x_s^j}{x_r^j} + \displaystyle\sum_{\substack{l\neq s\\ l\neq p}}^{m} \dfrac{x_p^j}{x_l^j}\right) & \text{para}\, f_{ji}(s) \neq r \end{cases}$$

These constructions extend to n-person games the studied done for bimatrix games. We might have expected that the procedure used here could also serve to generate games with unique equilibrium in Polymaric Games, however the technique we used didn't provide unique equilibrium in Polymaric Games and it would require the use of a different technique.

5.- Cyclic games and a model for the control of corruption evolution.

A dynamic model for analyzing corruption evolution was recently presented by Accinelli, Martins, Oviedo, Pinto and Quintas (2017). The aim of that article was to analyze the evolution of the corruption focusing in how to control the central authority ("the controller") to deter the expansion of the corruption. This role played by citizens is crucial and it is found that politically active citizens can prevent the spread of corruption. It is considered a game between government and the officials where both can choose between a corrupt or honest behavior. Citizens have a political influence that results in the prospects of a corrupt or non corrupt government can be reelected or not. It is modeled by mean of an index of intolerance to corruption. The evolutionary version of the game is given by means of the replicator dynamics.

The whole game can be viewed as a 3 person Cyclic game: the government plays a game with the officials designed to fulfill a public function. Both, the government and the officials can act in an honest or dishonest way. The officials play themselves a game with the citizens, having the possibility to ask and receive bribes from the citizens. The cycle is closed by the interaction between the government and citizens who can be more or less sensible to the corruption, and can decide to punish the governments by not reelecting them.

This model can be generalized to an n-person Cyclic game by mean of a top to bottom chain of two person games played by the government and the main official, this with the second level official, and so on with the lower levels officials. The cycle is closed again by the 2-person game played by the citizens and the government.

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