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Axiomatic solutions for network situations

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Abstract

In this work we analyse solutions for network situations, based on the basic properties of linearity and symmetry. Through such analysis, it is shown a relation ship between linear symmetric solutions and the representation of the group of permutations for the cases of three and four agents (nodes). Finally, additional properties are included in order to obtain axiomatic characterizations of certain classes of solutions.

Keywords: Network situations, axiomatic solutions, representation theory, symmetric group.

1 Introduction

The way in which people can cooperate (or organize) through networks is a point of interest in many applications. Examples of the effects of social networks on economic activity are abundant and pervasive, including roles in transmitting information about jobs, new products, technologies, and political opinions. A common denominator among these situations is that the way in which players are connected to each other is important in determining the total productivity or value generated by the group.

Myerson (1977) made a contribution in augmenting a cooperative game by a network structure specifying which groups of players can communicate and achieve their worth. The feasible groups are the ones whose members can communicate via the given network. This author showed that there exists an extension of the Shapley value (Shapley, 1953) to these kind of cooperative games, providing a simple characterization of it. This allocation rule has come to be called the Myerson value in the subsequent literature.

In a more general context, Jackson and Wolinsky (1996) have introduced a class of games – network games – where the value generated by a group of players depends directly on the network structure. They have extended the Myerson value to network games and study the stability and efficiency of social and economic networks, when selfinterested individuals can form or sever links.

More recently, Jackson (2005) take an axiomatic point of view for solving network games and presents a family of allocation rules that incorporate information about alternative network structures when allocating value.

In this article we study solutions for network games that satisfy the elementary properties of linearity and symmetry, for the cases of three and four players. This study presents the innovative use of basic representation theory[1] of the group of permutations of the set of players (symmetric group) and provides a different perspective than the more 'traditional' approaches.

Roughly speaking, representation theory is a general tool for studying abstract algebraic structures by representing their elements as linear transformations of vector spaces. It makes sense to use it,

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[1] A recommended reference for basic representation theory is Fulton and Harris (1991)

since every permutation may be thought of as a linear map[2] and it presents the information in a more clear and concise way.

In a few words, what we do is to compute direct sum decomposition of the space of network games (via the space of value functions) and the space of payoffs into "elementary pieces". According to this decomposition, any linear symmetric solution when restricted to any such elementary piece is either zero or multiplication by a single scalar; therefore, all linear symmetric solutions may be written as a sum of trivial maps.

With a global description of all linear and symmetric solutions, it is easy to understand the restrictions imposed by other conditions (e.g., the efficiency axiom). We then use such decomposition to provide, in a very economical way, a characterization for the class of linear symmetric solutions and the class of all linear, symmetric and efficient solutions.

The paper is organized as follows. We first recall the main basic features of network games and their solutions in the next section. A decomposition for the space of value functions with three and four players is introduced in section 3. In section 4 we show an application of this decomposition by giving characterizations of linear symmetric solutions and section 5 concludes the paper.

2 Preliminaries

Let $N = \{1, 2, ..., n\}$ be a fixed nonempty finite set, and let the members of N be interpreted as *players* (or nodes) who are connected in some network relationship.

A *network* is a list of which pairs of players are linked to each other and is modeled as a non-directed graph[3].

Definition 1 *A network g is a set of unordered pairs of players $\{i, j\}$, where $\{i, j\} \in g$ indicates that i and j are linked under the network g.*

When there is no place of confusion and for simplicity, we will write just ij to represent the link $\{i, j\}$. In this way, $ij \in g$ indicates that i and j are linked under the network g.

More formally, let g^N be the set of all subsets of N of size 2. In other words, g^N will denote the *complete network* where all the players are linked with each other.

The set of all possible networks or graphs on N will be denoted by G:

$$G = \{g \mid g \subseteq g^N\}$$

The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network obtained by deleting link ij from an existing network g is denoted $g - ij$.

For $g \in G$, let $N(g)$ be the set of players who have at least one link in g. That is, $N(g) = \{i \mid \exists j$ s.t. $ij \in g\}$. Let $n(g) = |N(g)|$ be the number of players involved in g.

Let $Li(g)$ be the set of links that player i is involved in, so that $Li(g) = \{ij \mid \exists j \text{ s.t. } ij \in g\}$, and let $\ell_i(g) = |Li(g)|$.

Given any subset (coalition) $S \subseteq N$, let g^S be the complete network among the players in S and let

$$g \mid_S = \{ij \mid ij \in g \text{ and } i, j \in S\}$$

Thus $g \mid_S$ is the network found deleting all links except those that are between players in S.

Remark 1 *Notice the distinction between the notation g^S which is the complete network among players in S, and $g \mid_S$ which is the network found by starting with some g and then eliminating links involving players outside of S.*

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[2] The precise statement will be provided in Sec. 3.

[3] That is, it is not possible for one individual to link to another, without having the second individual also linked to the first.

Definition 2 *A path in a network $g \in G$ between players i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.*

From the path relationships in a network, it can be naturally partitioned into different connected subgraphs that are commonly referred to as components.

Definition 3 *A component of a network g, is a non-empty subnetwork $g' \subseteq g$, such that*

a) *if $i, j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j*

b) *if $i \in N(g')$ and $ij \in g$, then $ij \in g'$*

In this way, the components of a network are the distinct connected subgraphs of a network. The set of components of g will be denoted by $C(g)$.

Notice that $g = \cup_{g' \in C(g)} g'$ and under this definition of component, a completely isolated player who has no links is not considered a component.

Also we need to define certain sets which are used in the sequel. For this purpose, given a network g with a fixed number of links k, we will say that a number l is conceivable (for k) if there exists a player such that $\ell_i(g) = l$.

Definition 4 *Let A_n be the set defined by*

$$A_n = \left\{ (k, l) \mid k \in \left\{ 1, \ldots, \binom{n}{2} \right\} \text{ and } l \text{ is conceivable for } k \right\}$$

It is of interest to know the total productivity of a graph and this notion is captured by a value function.

Definition 5 *A value function is a mapping*

$$\omega : G \to \mathbb{R}$$

such that $\omega(\varnothing) = 0$. The set of all possible value functions is denoted Γ, i.e.,

$$\Gamma = \{\omega : G \to \mathbb{R} \,|\omega(\varnothing) = 0\}$$

The number $\omega(g)$ specifies the total value that is generated by a given network structure g. The calculation of value may involve both costs and benefits and is a richer object than a characteristic function of a cooperative game, as it allows the value that accrues to depend on the network structure and not only on the coalition of players involved.

Given $\omega_1, \omega_2 \in \Gamma$ and $c \in \mathbb{R}$, we define the sum $\omega_1 + \omega_2$ and the product $\lambda\omega_1$, in Γ, in the usual form, i.e.,

$$(\omega_1 + \omega_2)(g) = \omega_1(g) + \omega_2(g) \quad \text{and} \quad (\lambda\omega_1)(g) = \lambda\omega_1(g)$$

respectively. It is easy to verify that Γ is a vector space (over $\mathbb{R}$) with these operations.

For subsequent analysis we will use the notation $G^{(n)}$ and $\Gamma^{(n)}$ to emphasize over a particular number n of players considered in the set G and on the space Γ, respectively.

An interesting sub-class of value functions are those where the value to a given component of a network does not depend on the structure of other components. This precludes externalities across (but not within) components of a network.

Definition 6 *A value function ω is component additive if for any $g \in G$:*

$$\sum_{g' \in C(g)} \omega(g') = \omega(g)$$

Definition 7 *A network game is a pair* (N, ω)*, where* N *is the set of players and* ω *is a value function.*

In order to know how the total productivity of a network (in a network game) is allocated among the individual nodes, we need to define the notion of a solution.

Definition 8 *A solution is a function*

$$\varphi : G \times \Gamma \to \mathbb{R}^n$$

where $\varphi_i(g, \omega)$ *is interpreted as the utility payoff which player* i *should expect from the network game* (N, ω) *for a fixed network* g.

The previous notion of a solution is the one that we will use for the analysis in this article. In the same sense, it is common to find in the literature the concept of an allocation rule.

Definition 9 *An allocation rule is a function* $\varphi : G \times \Gamma \to \mathbb{R}^n$ *such that*

$$\sum_{i \in N} \varphi_i(g, \omega) = \omega(g) \qquad \forall g \text{ and } \forall \omega \tag{1}$$

Remark 2 *Notice the difference between the concepts of solution and allocation rule. While a solution is a more general concept, an allocation rule is a more restrictive one: it is a solution that satisfies the condition imposed by (1), which stands for an efficiency-type property.*

Due to the richness of network games, several solutions (allocation rules) have been given for these problems. For example, as mentioned in the introduction, the first paper that proposed a value concept for network problems was Myerson (1977). It is an allocation rule that was defined in the context of cooperative games with communication structures, that is a variation on the Shapley value. The following presentation of the Myerson value is due to Jackson and Wolinsky (1996), which it is an easy extension of the main theorem of Myerson (1977). Such allocation rule satisfies the following axioms.

Axiom 1 (Component balance (CB)) *An allocation rule* φ *satisfies component balance if for any component additive* ω*,* $g \in G$*, and* $g' \in C(g)$

$$\sum_{i \in N(g')} \varphi_i(g, \omega) = \omega(g')$$

Component balance requires that if a value function is component additive, then the value generated by any component be allocated to the players among that component.

Axiom 2 (Equal bargaining power (EBP)) *An allocation rule* φ *satisfies equal bargaining power*[4] *if for any component additive* ω *and* $g \in G$

$$\varphi_i(g, \omega) - \varphi_i(g - ij, \omega) = \varphi_j(g, \omega) - \varphi_j(g - ij, \omega)$$

This axiom does not requires that players split the marginal value of a link; instead, it just requires that they equally benefit or suffer from its addition.

Theorem 1 (Jackson and Wolinsky, 1996) *There exists a unique allocation rule* ψ^M *that satisfies CB and EBP. Moreover, it is given by*

$$\psi_i^M(g, \omega) = \sum_{S \subseteq N_{-i}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left[\omega(g\mid_{S_{+i}}) - \omega(g\mid_S)\right] \tag{2}$$

for all $g \in G$ *and any component additive* ω*.*

[4] This was called *fairness* by Myerson (1977).

2.1 The basic properties

For the study of solutions of network games using representation theory techniches, the reasonable requirements that are necessary to impose are the usual linearity and symmetry axioms. These axioms will be a key ingredient in subsequent developments. Next, we define them.

First, the group of permutations of N, $\Pi_n = \{\pi : N \to N \mid \pi \text{ is bijective}\}$, acts on G (set of networks) as well as on $\mathbb{R}^n$ (space of payoff vectors) in a natural way; i.e.,

- For $g \in G$ and $\pi \in \Pi_n$:
$$\pi(g) = \{\pi(i)\pi(j) \mid ij \in g\}$$
- For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $\pi \in \Pi_n$:
$$\pi \cdot (x_1, x_2, ..., x_n) = (x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$$

Moreover, the group Π_n acts on the space of value functions, Γ, in the following way. If $\omega \in \Gamma$ and $\pi \in \Pi_n$, then
$$[\pi \cdot \omega](g) = \omega\left[\pi^{-1}(g)\right]$$

Now, the formal restrictions are the following.

Axiom 3 (Linearity) *The solution φ is linear if for every $g \in G$, every $\omega_1, \omega_2 \in \Gamma$ and every $c \in \mathbb{R}$:*
$$\varphi(g, c \cdot \omega_1 + \omega_2) = c \cdot \varphi(g, \omega_1) + \varphi(g, \omega_2)$$

Axiom 4 (Symmetry) *The solution φ is said to be symmetric if and only if*
$$\varphi(\pi(g), \pi \cdot \omega) = \pi \cdot \varphi(g, \omega)$$
for every $\pi \in \Pi_n$, every $g \in G$ and every $\omega \in \Gamma$.

The axiom of linearity means that in the sharing of benefits (or costs) stemming from two different issues, how much each player obtains does not depend on whether they consider the two issues together or one by one. Hence, the agenda does not affect the final outcome. Also, the sharing does not depend on the unit used to measure the benefits.

Whereas, the symmetry axiom means that player's payoffs do not depend on their names and it is only derived from his influence on the value of the networks. The axiom requires that if all that has changed is the labels of the players and the value generated by networks has changed in an exactly corresponding fashion, then the allocation only change according to the relabeling

Remark 3 *It is not difficult to show that the Myerson value is a solution that satisfies the properties of linearity and symmetry.*

2.2 Group action of Π_n

The symmetric group Π_n acts on Γ via linear transformations (i.e., Γ is a representation of Π_n). That is, there is a group homomorphism $\rho : \Pi_n \to GL(\Gamma)$, where $GL(\Gamma)$ is the group of invertible linear maps in Γ. This action is given by:
$$(\pi \cdot \omega)(g) := [\rho(\pi)(\omega)](g) = \omega[\pi^{-1}(g)]$$
for every $\pi \in \Pi_n$, $\omega \in \Gamma$ and $g \in G$.

Moreover, this assignment preserves multiplication (i.e., is a group homomorphism) in the sense that the linear map corresponding to the product of the two permutations $\pi_1\pi_2$ is the product (or composition) of the maps corresponding to π_1 and π_2, in that order.

Similarly, the space of payoff vectors, $\mathbb{R}^n$, is a representation for Π_n:
$$\pi \cdot (x_1, x_2, ..., x_n) = (x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$$

Definition 10 *Let X_1 and X_2 be two representations for the group Π_n. A linear map $T : X_1 \to X_2$ is said to be Π_n-equivariant if $T(\pi \cdot x) = \pi \cdot T(x)$, for every $\pi \in \Pi_n$ and every $x \in X_1$.*

Definition 11 *Let Y be a subspace of a vector space X.*

- *Y is invariant (for the action of Π_n) if for every $y \in Y$ and every $\pi \in \Pi_n$, we have that*

$$\pi \cdot y \in Y$$

- *Y is irreducible if Y itself has no invariant subspaces other than $\{0\}$ and Y itself.*

Remark 4 *Notice that, what we are calling a linear symmetric solution (for network games) ,in the language of representation theory means a linear map that is Π_n-equivariant.*

3 Decompositions

In order to study linear symmetric solutions (on Γ) by taking advantage of the group action of Π_n on Γ and on $\mathbb{R}^n$, it is necessary to obtain a decomposition of these spaces into irreducible representations.

We begin with the decomposition of $\mathbb{R}^n$ into irreducible representations, which is easier, and then proceed to do the same thing for Γ; that is, we wish to write $\mathbb{R}^n$ as a direct sum of subspaces, each invariant for all permutations in Π_n and in such way that the summands cannot be further decomposed (i.e., they are irreducible).

For this, let

$$U_n = \{(t, t, \ldots, t) \in \mathbb{R}^n \mid t \in \mathbb{R}\} \quad \text{and} \quad V_n = U_n^{\perp} = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 0 \right\}$$

The spaces U_n and V_n are usually called the "trivial" and "standard" representations, respectively. Notice that U_n is a trivial subspace in the sense that every permutation acts as the identity transformation.

Every permutation fixes every element of U_n, so, in particular, it is an invariant subspace of $\mathbb{R}^n$. Being 1-dimensional, it is automatically irreducible. Its orthogonal complement, V_n, consists of all vectors such that the sum of their coordinates is zero. Clearly, if we permute the coordinates of any such vector, its sum will still be zero. Hence V_n is also an invariant subspace.

Proposition 1 *The decomposition of $\mathbb{R}^n$, under Π_n, into irreducible subspaces is:*

$$\mathbb{R}^n = U_n \oplus V_n$$

Proof. First, it is clear that $U_n \cap V_n = \{(0, \ldots, 0)\}$. We now prove that $\mathbb{R}^n = U_n + V_n$:

i) If $z \in (U_n + V_n)$, then $z \in \mathbb{R}^n$ since $(U_n + V_n)$ is a subspace of $\mathbb{R}^n$.

ii) For $z \in \mathbb{R}^n$, let $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ and z can be written as $z = (\overline{z}, \overline{z}, ..., \overline{z}) + (z_1 - \overline{z}, z_2 - \overline{z}, ..., z_n - \overline{z})$; and so, $z \in (U_n + V_n)$.

Now, since U_n is 1-dimensional, then it is irreducible and to check that V_n is also irreducible, it is enough to use an induction argument. ■

In this way, this result tell us that $\mathbb{R}^n$ as a vector space with group of symmetry Π_n, can be written as an orthogonal sum of the subspaces U_n and V_n, which are invariant under permutations and which can no longer be further decomposed.

The decomposition of Γ is carried out in several steps. First, we establish a partition (into distinct classes) of the set of networks in the following way.

Definition 12 *Let $g_1, g_2 \in G \backslash \{\varnothing\}$, we will say that g_1 and g_2 belong to the same class if $\exists \pi \in \Pi_n$ such that $\pi(g_1) = g_2$.*

Let $m_G \in \mathbb{N}$ be the number of different classes in which the set $G \backslash \{\varnothing\}$ can be partitioned according to Definition (12). Thus, if G_j denotes the set of networks that belong to the class j, then:

$$G \backslash \{\varnothing\} = \bigcup_{j=1}^{m_G} G_j$$

where we can notice that $G_j \cap G_k = \varnothing$ if $j \neq k$.

For further analysis, we will asume that G_1 is the class of networks with exactly 1 link and G_{m_G} is the class of networks that contains the complete network; i.e.,

$$G_1 = \{g \in G \mid |g| = 1\} \quad \text{and} \quad G_{m_G} = \{g^N\}$$

Now we turn back to the decomposition of Γ. For each $k \in \{1, \ldots, m_G\}$, define the subspace of value functions

$$\Gamma_k = \{\omega \in \Gamma \mid \omega(g) = 0 \text{ if } g \notin G_k\} \tag{3}$$

Then the space Γ has the following decomposition:

$$\Gamma = \bigoplus_{k=1}^{m_G} \Gamma_k \tag{4}$$

Each subspace Γ_k is invariant under Π_n and the decomposition is orthogonal with respect to the invariant inner product on Γ given by

$$\langle \omega_1, \omega_2 \rangle = \sum_{g \in G} \omega_1(g) \cdot \omega_2(g) \tag{5}$$

Here, invariance of the inner product means that every permutation $\pi \in \Pi_n$ is not only a linear map on Γ, but an orthogonal map with respect to this inner product. Formally, $\langle \pi \cdot \omega_1, \pi \cdot \omega_2 \rangle = \langle \omega_1, \omega_2 \rangle$ for every $\omega_1, \omega_2 \in \Gamma$.

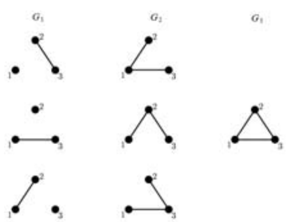
3.1 The case $n = 3$

According to the previous definitions, for the set of networks of $n = 3$ nodes, $G^{(3)}$, it turns out that $m_{G^{(3)}} = 3$ and these classes are given by

$$G_1^{(3)} = \{\{12\}, \{13\}, \{23\}\}$$

$$G_2^{(3)} = \{\{12, 13\}, \{12, 23\}, \{13, 23\}\}$$

$$G_3^{(3)} = \{\{12, 13, 23\}\}$$



and according to (3), the space of value functions is decomposed as

$$\Gamma^{(3)} = \Gamma_1^{(3)} \oplus \Gamma_2^{(3)} \oplus \Gamma_3^{(3)}$$

The next goal is to get a decomposition of each subspace of value functions $\Gamma_k^{(3)}$ into irreducible subspaces and so, we will get it for $\Gamma^{(3)}$.

The following value functions play an important role in describing the decomposition of the space Γ. For $k \in \{1, \ldots, m_G\}$, define $c_k \in \Gamma_k$ as follows

$$c_k(g) = \begin{cases} 1 & \text{if } g \in G_k \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Note that $\Gamma_{m_G} = \mathbb{R}c_{m_G}$.

Also, for each $k \in \{1, \ldots, m_G\}$ and for each $z \in \mathbb{R}^n$; define the value function $z^k \in \Gamma_k$ as follows

$$z^k(g) = \begin{cases} \sum\limits_{ij \in g} (z_i + z_j) & \text{if } g \in G_k \\ 0 & \text{otherwise} \end{cases}$$

Definition 13 *Suppose X_1 and X_2 are two representations for the group Π_n, i.e., we have two vector spaces X_1 and X_2 where Π_n is acting by linear maps. We say that X_1 and X_2 are isomorphic if there is a linear map between them, which is $1-1$ and onto and that commutes with the respective Π_n-actions. Formally, there is an invertible linear map $T : X_1 \to X_2$ such that $T(\pi \cdot x) = \pi \cdot T(x)$, for every $\pi \in \Pi_n$ and every $x \in X_1$. We then write $X_1 \simeq X_2$.*

For our purposes, X_1 will be an irreducible subspace of Γ and X_2 an irreducible subspace of $\mathbb{R}^n$.

Isomorphic representations are essentially "equal"; not only are they spaces of the same dimension, but the actions are equivalent under some linear invertible map between them.

The next Proposition provides a decomposition of the space of value functions for $n = 3$ players (nodes), into irreducible subspaces.

Proposition 2 *For $k \in \{1, 2\}$,*

$$\Gamma_k^{(3)} = C_k^{(3)} \oplus R_k^{(3)}$$

where $C_k^{(3)} = \langle c_k \rangle \simeq U_3$ and $R_k^{(3)} = \left\{ z^k \mid z \in V_3 \right\} \simeq V_3$. The decomposition is orthogonal.

Proof. *Since for an arbitrary representation X of S_3, one can write*

$$X = U_3^{\oplus a} \oplus U'^{\oplus b} \oplus V_3^{\oplus c} \tag{7}$$

and there is a way to determine the multiplicities a, b and c; in terms of $\tau = (123)$ and $\sigma = (12)$, which generates S_3. c for example, is the number of independent eigenvectors for τ with eigenvalue ε (denoting by $1, \varepsilon, \varepsilon^2$ the cube roots of unity), whereas $a + c$ is the multiplicity of 1 as an eigenvalue of σ, and $b + c$ is the multiplicity of -1 as an eigenvalue of σ.

In this way we start by showing that $\Gamma_k^{(3)}$ has exactly 1 copy of U_3 and 1 copy of V_3, if $k \in \{1, 2\}$. It is clear that $\mathfrak{B} = \{\omega_{\widetilde{g}} \mid \widetilde{g} \in G \backslash \{\varnothing\}\}$ form a basis for $\Gamma^{(3)}$, where

$$\omega_{\widetilde{g}}(g) = \begin{cases} 1 & \textit{if } g = \widetilde{g} \\ 0 & \textit{otherwise} \end{cases} \tag{8}$$

For $\Gamma^{(3)}$, it is easy to verify that $[\tau]_{\mathfrak{B}}$ has the characteristic polynomial $p(x) = \left[(x-1)(x-\varepsilon)(x-\varepsilon^2)\right]^2 (x-1)$ and $[\sigma]_{\mathfrak{B}}$ has the characteristic polynomial $p(x) = (x+1)^2(x-1)^5$. From these and (7), we have that $c = 2$, $a + c = 5$ and $b + c = 2$. Then

$$\Gamma^{(3)} = U_3^{\oplus 3} \oplus V_3^{\oplus 2}$$

This implies directly that if $k \in \{1, 2\}$, then every Γ_k has exactly 1 copy of U_3 and 1 copy of V_3, since $\Gamma_{m_G} = \mathbb{R}c_{m_G} \simeq U_3$ and $\dim \Gamma_k = 3$.

Now, define the map $T^k : \mathbb{R}^n \to \Gamma_k$ by $T^k(z) = z^k$. This map is an isomorphism between $C_k^{(3)}$ and U_3 (similarly, between $R_k^{(3)}$ and V_3) since it is linear, S_3-equivariant and $1-1$. From Proposition 1 we have the splitting $\mathbb{R}^3 = U_3 \oplus V_3$. Thus, inside of Γ_k, we have the images of these two subspaces: $C_k^{(3)} = T^k(U_3)$ and $R_k^{(3)} = T^k(V_3)$.

Finally, the invariant inner product $\langle , \rangle$ gives an equivariant isomorphism, in particular must preserve the decomposition. This implies orthogonality of the decomposition. ■

Remark 5 *Recall that* $\Gamma_3^{(3)}$ *is a trivial representation generated by the value function that assigns* 1 *to the complete network and* 0 *elsewhere.*

Whereas from the above Proposition, it is not difficult to verify that for $k \in \{1, 2\}$*:*

$$C_k^{(3)} = \{\omega \in \Gamma_k^{(3)} \mid \omega(g_1) = \omega(g_2) \text{ if } |g_1| = |g_2|\}$$

and

$$R_k^{(3)} = \left\{ \omega \in \Gamma_k^{(3)} \mid \sum_{\{g \in G : |g| = k\}} \omega(g) = 0 \right\}$$

Proposition 2 gives a decomposition of the space of games that is a key ingredient in our subsequent analysis.

Set $C^{(3)} = C_1^{(3)} \oplus C_2^{(3)} \oplus C_3^{(3)}$. This is a subspace of value functions whose value on a given network g, depends only on the number of links that form such network. According to Proposition 2, $C^{(3)}$ is the largest subspace of $\Gamma^{(3)}$ where S_3 acts trivially[5]. Let $R^{(3)} = R_1^{(3)} \oplus R_2^{(3)}$. Then

$$\Gamma^{(3)} = C^{(3)} \oplus R^{(3)}$$

Thus, given a value function $\omega \in \Gamma^{(3)}$ we may decompose it relative to the above as $\omega = c + r$, where in turn $u = \sum a_k c_k$ and $r = \sum z_k^k$. This decomposition is very well suited to study the image of ω under any linear symmetric solution. The reason being the following version of the well known Schur's Lemma[6].

Theorem 2 (Schur's Lemma) *Any linear symmetric solution*

$$\varphi : G^{(3)} \times \Gamma^{(3)} = G^{(3)} \times \left[C^{(3)} \oplus R^{(3)}\right] \to \mathbb{R}^3 = U_3 \oplus V_3$$

satisfies

a) $\varphi\left[G^{(3)} \times C^{(3)}\right] \subset U_3$

b) $\varphi\left[G^{(3)} \times R^{(3)}\right] \subset V_3$

Moreover,

- *for each* $k \in \{1, 2, 3\}$*, there is a constant* $\exists \alpha_k \in \mathbb{R}$ *such that for every* $(g, \omega) \in G^{(3)} \times C_k^{(3)}$*,*

$$\varphi(g, \omega) = \alpha_k (1, 1, 1) \in U_3$$

- *for each* $k \in \{1, 2\}$*, there is a constant* $\beta_k \in \mathbb{R}$ *such that for every* $\left(g, z^k\right) \in G^{(3)} \times R_k^{(3)}$*,*

$$\varphi(g, z^k) = \beta_k z \in V_3$$

For many purposes it suffices to use merely the existence of the decomposition of the value function $\omega \in \Gamma^{(3)}$, without having to worry about the precise value of each component. Nevertheless it will be useful to have it. Thus we give a formula for computing it.

Proposition 3 *Let* $\omega \in \Gamma^{(3)}$*. Then*

$$\omega = \sum_{k=1}^{3} a_k c_k + \sum_{k=1}^{2} z_k^k \tag{9}$$

where,

[5] i.e., $\theta \cdot \omega = \omega$ for every $\theta \in S_3$ and every $\omega \in C^{(3)}$.

[6] See the Appendix for a precise statement.

i) *a_k is the average of the values $\omega(g)$ with $|g| = k$:*

$$a_k = \frac{\sum_{|g|=k} \omega(g)}{|\{g \in G \mid |g| = k\}|}$$

ii) *For every $k \in \{1, 2\}$:*

$$(z_k)_i = \sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)\neq k}} (3-k)\omega(g)$$

Proof. *We start by computing the orthogonal projection of ω onto $C^{(3)}$. Notice that $\{c_k\}$ is an orthogonal basis for $C^{(3)}$, and that $\|c_k\|^2 = \left|\{g \in G^{(3)} \mid |g| = k\}\right|$.*

Thus, the projection of ω onto $C^{(3)}$ is

$$\sum_{k=1}^{3} \frac{\langle \omega, c_k \rangle}{\langle c_k, c_k \rangle} c_k$$

and so,

$$a_k = \frac{\langle \omega, c_k \rangle}{\langle c_k, c_k \rangle} = \frac{\sum_{|g|=k} \omega(g)}{\left|\{g \in G^{(3)} \mid |g| = k\}\right|}$$

Now, for each $k \in \{1, 2, 3\}$, we define $h^k : \Gamma^{(3)} \to \mathbb{R}^3$ as

$$h_i^k(\omega) = \sum_{\substack{|g|=k \\ \ell_i(g)=k}} \omega(g)$$

where each h^k is S_3-equivariant and observe that $h^3(\omega) = \omega\left(g^N\right)(1, 1, 1)$. Let $z \in V_3$, then $h^k\left(z^l\right) = 0$ if $k \neq l$, whereas (by Schur's Lemma) for $k \in \{1, 2, 3\}$, $\exists \lambda_k \in \mathbb{R}$ such that $h^k(z^k) = \lambda_k z$.

Let $p : \mathbb{R}^3 \to V_3$ be the projection of $\mathbb{R}^3$ onto V_3 given by

$$p_i(x) = x_i - \overline{x}$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. This projection is equivariant, sends U_3 to zero and it is the identity on V_3.

Since $\left(p \circ h^k\right)(\omega) = \lambda_k z_k$, then $z_k = \frac{1}{\lambda_k} p\left(h^k(\omega)\right)$. Thus, we evaluate

$$\begin{aligned} p\left(h^k(\omega)\right) &= \lambda_k z_k \\ &= \sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)\neq k}} (3-k)\omega(g) \end{aligned}$$

■

3.2 The case $n = 4$

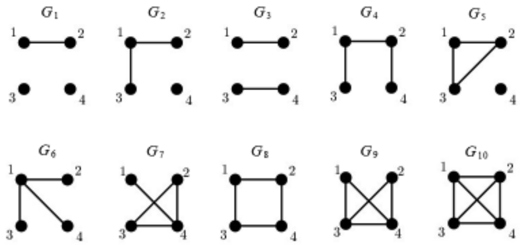
As we have noticed, all previous applications and results follow from the decomposition of the space of value functions into direct sum of irreducible subspaces. In this part, we provide such decomposition for the particular case of four players.

In the case of three players, the set $G^{(3)}$ was partitioned into 3 classes (the jth class contains networks with exactly j links, for $j \in \{1, 2, 3\}$). However, the partition of $G^{(4)}$ does not follow the same line of reasoning. The next example shows that there are networks with the same number of links, however they belong to different classes (recall Definition 12).

Example 1 *Let $N = \{1, 2, 3, 4\}$.*

- *The networks $g_1 = \{12, 13, 24\}$ and $g_2 = \{12, 24, 34\}$ belong to the same class, since there is a permutation $\pi \in S_4$ such that $\pi(g_1) = g_2$. Such a permutation is given by $\pi(1) = 2$, $\pi(2) = 4$, $\pi(3) = 1$ and $\pi(4) = 3$.*
- *The networks $g_1 = \{24, 34\}$ and $g_2 = \{12, 34\}$ do not belong to the same class, since $\nexists \pi \in S_4$ such that $\pi(g_1) = g_2$.*

Notice that $\left|G^{(4)} \setminus \{\varnothing\}\right| = 63$ and according to Definition 12, $m_{G^{(4)}} = 10$ classes. The following networks are representatives of each class.



Whereas the number of networks belonging to each class is shown below.

k	1	2	3	4	5	6	7	8	9	10
$\left\|G_k^{(4)}\right\|$	6	12	3	12	4	4	12	3	6	1

We follow the same line of reasoning as before, i.e., we first obtain a decomposition of each subspace of value functions $\Gamma_k^{(4)}$ into irreducible subspaces and so, we will get it for $\Gamma^{(4)}$.

For that purpose, let $z \in V_4$ and for $k \in \{1, 4, 5, 6, 9\}$ define the value functions $z^k \in \Gamma_k^{(4)}$ as

$$z^k(g) = \begin{cases} \sum\limits_{ij \in g} (z_i + z_j) & \text{if } g \in G_k^{(4)} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

also define $z^2, z^{2'} \in \Gamma_2^{(4)}$ and $z^7, z^{7'} \in \Gamma_7^{(4)}$, as

$$z^2(g) = \begin{cases} \sum\limits_{ij \in g} (z_i + z_j) & \text{if } g \in G_2^{(4)} \text{ and } \ell_1(g) = 1 \\ 0 & \text{otherwise} \end{cases} ; \quad z^{2'}(g) = \begin{cases} \sum\limits_{ij \in g} (z_i + z_j) & \text{if } g \in G_2^{(4)} \text{ and } \ell_1(g) \neq 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

$$z^7(g) = \begin{cases} \sum\limits_{ij \in g} (z_i + z_j) & \text{if } g \in G_7^{(4)} \text{ and } \ell_1(g) = 2 \\ 0 & \text{otherwise} \end{cases} ; \quad z^{7'}(g) = \begin{cases} \sum\limits_{ij \in g} (z_i + z_j) & \text{if } g \in G_7^{(4)} \text{ and } \ell_1(g) \neq 2 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

The nature of the previous value functions will be justified in the decomposition of $\Gamma^{(4)}$, presented in the following

Proposition 4 *For $k \in \{1, \ldots, 9\}$, the decomposition of each $\Gamma_k^{(4)}$ (under S_4) into irreducible subspaces is:*

$$\Gamma_k^{(4)} = C_k^{(4)} \oplus R_k^{(4)} \oplus T_k^{(4)}$$

where,

- $C_k^{(4)} = \langle c_k \rangle \simeq U_4$ *if* $k \in \{1, \ldots, 9\}$

- $R_k^{(4)} = \begin{cases} \{z^k \mid z \in V_4\} \simeq V_4 & \text{if } k \in \{1,4,5,6,9\} \\ \{z^k \mid z \in V_4\} \cup \left\{z^{k'} \mid z \in V_4\right\} & \text{if } k \in \{2,7\} \end{cases}$

- $T_k^{(4)} = \left(C_k^{(4)} \oplus R_k^{(4)}\right)^\perp$ *does not contain any summands isomorphic to either* U_4 *nor* V_4.

The decomposition is orthogonal.

Proof. *The proof is based on the use of character theory, which it is a remarkably effective technique for decomposing any given finite dimensional representation into its irreducible components. Recall that if* $\rho : H \to GL(X)$ *is any representation, the character of* X *is the complex-valued function* $\chi_X : H \to \mathbb{C}$, *defined as* $\chi_X(h) = Tr(\rho(h))$. *For* $\chi_1, \chi_2 \in \mathbb{C}_{class}(H)$, *it is defined an Hermitian inner product on* $\mathbb{C}_{class}(H)$ *by*

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_1(h)} \cdot \chi_2(h) \tag{13}$$

This inner product allows to calculate the multiplicities of irreducible subspaces in a representation. For instance, if $Z = Z_1^{\oplus a_1} \oplus Z_2^{\oplus a_2} \oplus \cdots \oplus Z_j^{\oplus a_j}$, *then the multiplicity* Z_i *(irreducible representation) in* Z, *is given by* $a_i = \langle \chi_Z, \chi_{Z_i} \rangle$, *where* $\langle , \rangle$ *is the inner product given by (13).*

Then, $\left\langle \chi_{\Gamma_k^{(4)}}, \chi_{U_4} \right\rangle$ *and* $\left\langle \chi_{\Gamma_k^{(4)}}, \chi_{V_4} \right\rangle$ *are the number of subspaces isomorphic to the trivial (*U_4*) and standard representation (*V_4*) within* $\Gamma_k^{(4)}$, *respectively. The characters for each* $\Gamma_k^{(4)}$ *are given by*[7]*:*

S_4	1 [(1)]	6 [(12)]	8 [(123)]	6 [(1234)]	3 [(12)(34)]
$\Gamma_1^{(4)}, \Gamma_9^{(4)}$	6	2	0	0	2
$\Gamma_2^{(4)}, \Gamma_7^{(4)}$	12	2	0	0	0
$\Gamma_3^{(4)}, \Gamma_8^{(4)}$	3	1	0	1	3
$\Gamma_4^{(4)}$	12	0	0	0	4
$\Gamma_5^{(4)}, \Gamma_6^{(4)}$	4	2	1	0	0
$\Gamma_{10}^{(4)}$	1	1	1	1	1

Thus from (13), $\left\langle \chi_{\Gamma_k^{(4)}}, \chi_{U_4} \right\rangle = 1$ *for each* $k \in \{1, \ldots, 10\}$ *and*

$$\left\langle \chi_{\Gamma_1^{(4)}}, \chi_{V_4} \right\rangle = \begin{cases} 1 & \text{if } k \in \{1,4,5,6,9\} \\ 2 & \text{if } k \in \{2,7\} \\ 0 & \text{if } k \in \{3,8,10\} \end{cases}$$

The last part is to identify such copies of U_4 *and* V_4 *inside* $\Gamma_k^{(4)}$. *To this end, for* $k \in \{1, \ldots, 10\}$ *let* $f_k : U_4 \to C_k^{(4)}$ *be given by* $f_k(u) = \omega_k$, *in which there exists* $t \in \mathbb{R}$ *such that* $u = t(1,1,1,1)$ *and* $\omega_k(g) = t$ *if* $g \in G_k$ *and* $\omega_k(g) = 0$ *otherwise. The function* f_k *is an isomorphism between* U_4 *and* $C_k^{(4)}$ *since it is linear,* S_4*-equivariant and one to one. Thus,* $\Gamma_k^{(4)}$ *contain the image of this subspace:* $C_k^{(4)} = f_k(U_4)$.

Now, for $k \in \{1,4,5,6,9\}$ *define the functions* $L_k : V^4 \to \Gamma_k^{(4)}$ *by* $L_k(z) = z^k$ *(given by (10)). These maps are isomorphisms between* $R_k^{(4)}$ *and* V_4, *and* $R_k^{(4)} = L_k(V_4)$.

In the same way, for $k \in \{2,7\}$ *define the functions* $L_k, L_{k'} : V^4 \to \Gamma_k^{(4)}$ *by* $L_k(z) = z^k$ *and* $L_{k'}(z) = z^{k'}$ *(given by (11) and (12)), respectively. Thus,* $R_k^{(4)} = L_k(V_4) \cup L_{k'}(V_4)$.

Orthogonality of the decomposition follows again from the fact that the invariant inner product $\langle , \rangle$ *gives an equivariant isomorphism, which preserves the decomposition.* ■

[7] In which a convenient basis is the one given in (8).

It is not difficult to verify that $\Gamma^{(4)}_{10} = \langle c_{10}\rangle \simeq U_4$ is a trivial representation generated by the value function that assigns 1 to the complete network and 0 elsewhere.

On the other hand, from the above Proposition it turns out that for $k \in \{1, \ldots, 9\}$:

$$C^{(4)}_k = \{\omega \in \Gamma^{(4)}_k \mid \omega(g_1) = \omega(g_2) \text{ if } g_1, g_2 \in G^{(4)}_k\}$$

Remark 6 *Proposition 4 does not quite give a decomposition of $\Gamma^{(4)}_k$ into irreducible summands. The subspace $C^{(4)}_k$ is irreducible and $R^{(4)}_k$ is a direct sum of irreducible subspaces. Whereas $T^{(4)}_k$ may or may not be irreducible (depending on k), but as we shall see the exact nature of this subspace plays no role in the study of linear symmetric solutions since it lies in the kernel of any such solution.*

Set $C^{(4)} = \bigoplus_{k=1}^{10} C^{(4)}_k$. This is a subspace of value functions whose value on a given network g, depends only on the "shape" of such network[8]. Let $R^{(4)} = \bigoplus_{k\in\{1,2,4,5,6,7,9\}} R^{(4)}_k$ and $T^{(4)} = \bigoplus_{k=1}^{10} T^{(4)}_k$. Then

$$\Gamma^{(4)} = C^{(4)} \oplus R^{(4)} \oplus T^{(4)}$$

Corollary 1 *If $\varphi : G^{(4)} \times \Gamma^{(4)} \to \mathbb{R}^4$ is a linear symmetric solution, then for every $(g, \omega) \in G^{(4)} \times T^{(4)}$:*

$$\varphi(g, \omega) = 0$$

Proof. Let $\varphi : G^{(4)} \times \Gamma^{(4)} = G^{(4)} \times \left[C^{(4)} \oplus R^{(4)} \oplus T^{(4)}\right] \to \mathbb{R}^4 = U_4 \oplus V_4$ be a linear symmetric solution. Assume $X \subset T^{(4)}$ is an irreducible summand in the decomposition of $T^{(4)}$ (even while we do not know the decomposition of $T^{(4)}$ as a sum of irreducible subspaces, it is known that such a decomposition exists). Let p_1 and p_2 denote orthogonal projection of $\mathbb{R}^4$ onto U_4 and V_4, respectively. Now, $\varphi : G^{(4)} \times \Gamma^{(4)} \to \mathbb{R}^4 = U_4 \oplus V_4$ maybe written as $\varphi = (p_1 \circ \varphi, p_2 \circ \varphi)$. Denote by $\iota : X \to G^{(4)} \times \Gamma^{(4)}$ the inclusion, then, the restriction of φ to X may be expressed as $\varphi_{|X} = \varphi \circ \iota = (p_1 \circ \varphi \circ \iota, p_2 \circ \varphi \circ \iota)$.

On the other hand, $p_1 \circ \varphi \circ \iota : X \to U_4$ and $p_2 \circ \varphi \circ \iota : X \to V_4$ are linear symmetric maps; since X is not isomorphic to either of these two spaces, thus Schur's Lemma (see Appendix for the statement) implies that $p_1 \circ \varphi \circ \iota$ and $p_2 \circ \varphi \circ \iota$ must be zero. Since this is true for every irreducible summand X of $T^{(4)}$, φ is zero on all of $T^{(4)}$. ■

Remark 7 *According to Proposition 4 and Corollary 1, in order to study linear symmetric solutions, one needs to look only at those value functions inside $C^{(4)} \oplus R^{(4)}$ (i.e., one have to take care of those copies of U_4 and V_4, contained in $\Gamma^{(4)}$).*

4 Characterization of solutions

In this section we show how to obtain characterizations of solutions (for the case of $n = 3$ nodes) easily by using the decomposition of a value function given by (9) in conjunction with Schur's Lemma. In a similar manner one could get these characterizations for the case $n = 4$.

We start by providing a characterization of all linear symmetric solutions $\varphi : G^{(3)} \times \Gamma^{(3)} \to \mathbb{R}^3$ in the following

Proposition 5 *The linear symmetric solutions $\varphi : G^{(3)} \times \Gamma^{(3)} \to \mathbb{R}^3$ are precisely those of the form*

$$\varphi_i(g, \omega) = \sum_{\substack{(k,l)\in A_3 \\ l\neq 0}} \sum_{\substack{|g|=k \\ \ell_i(g)=l}} \gamma_{(k,l)} \cdot \omega(g) + \sum_{(k,0)\in A_3} \sum_{|g|=k} \gamma_{(k,0)} \cdot \omega(g) \tag{14}$$

for some real numbers $\left\{\gamma_{(k,l)} \mid (k,l) \in A_3\right\}$.

[8] Following Jackson and Wolinsky (1996), the value functions in $C^{(4)}$ are known as "anonymous".

Proof. *Let $\varphi : G^{(3)} \times \Gamma^{(3)} \to \mathbb{R}^3$ be a linear symmetric solution. According to Proposition 3, $\omega \in \Gamma^{(3)}$ decomposes as*

$$\omega = \sum_{k=1}^{3} a_k c_k + \sum_{k=1}^{2} z_k^k$$

where by linearity,

$$\varphi_i(g,\omega) = \sum_{k=1}^{3} a_k \varphi_i(g, c_k) + \sum_{k=1}^{2} \varphi_i\left(g, z_k^k\right)$$

Now, from Schur's Lemma and Proposition 3, we have

$$\begin{aligned}\varphi_i(g,\omega) &= \sum_{k=1}^{3} \alpha_k a_k + \sum_{k=1}^{2} \beta_k (z_k)_i \\ &= \sum_{k=1}^{3} \alpha_k \frac{\sum_{|g|=k} \omega(g)}{|\{g \in G \mid |g| = k\}|} + \sum_{k=1}^{2} \beta_k \left[\sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)\neq k}} (3-k)\omega(g) \right]\end{aligned}$$

Finally, the result follows from grouping terms and by setting $\gamma_{(1,0)} = \frac{\alpha_1}{3} - 2\beta_1$, $\gamma_{(1,1)} = \frac{\alpha_1}{3} + \beta_1$, $\gamma_{(2,1)} = \frac{\alpha_2}{3} - \beta_2$, $\gamma_{(2,2)} = \frac{\alpha_2}{3} + 2\beta_2$ and $\gamma_{(3,2)} = \alpha_3$. ■

Corollary 2 *The space of all linear and symmetric solutions on $G^{(3)} \times \Gamma^{(3)}$ has dimension $|A_3| = 5$.*

Once we have such a global description of all linear symmetric solutions, we can understand restrictions imposed by other conditions or axioms. For example, we can consider that if all players decide to form the complete network (there is a link between any pair of players), then the value $\omega\left(g^N\right)$ is allocated among all the players. Formally:

Axiom 5 (Efficiency) *The solution φ is efficient if and only if for every $\omega \in \Gamma$:*

$$\sum_{i \in N} \varphi_i(g^N, \omega) = \omega\left(g^N\right)$$

Notice that, any allocation rule satisfies the efficiency axiom since it is the condition (1) restricted to g^N.

From the point of view of representation theory, the efficiency axiom has the following implications.

Proposition 6 *Let $\varphi : G^{(3)} \times \Gamma^{(3)} \to \mathbb{R}^3$ be a linear symmetric solution. Then φ is efficient if and only if*

i) *$\varphi_i(g^N, c_k) = 0$ for $k \in \{1,2\}$; and*

ii) *$\varphi_i(g^N, c_3) = \frac{1}{3}$*

Proof. First of all, $\left(C_3^{(3)}\right)^{\perp}$ is exactly the subspace of value functions ω where $\omega\left(g^N\right) = 0$. Of these, those in $R^{(3)}$ trivially satisfy $\sum_{i\in N} \varphi_i(g^N, \omega) = 0$, since (by Schur's Lemma) $\varphi\left(G \times R^{(3)}\right) \subset V_3$.

Thus, efficiency need only be checked in $C^{(3)}$. Since c_k is fixed by every permutation in S_3, we have

$$\sum_{i \in N} \varphi_i(g^N, c_k) = 3\varphi_i(g^N, c_k)$$

so, φ is efficient if and only if for $k \in \{1,2\}$,

$$3\varphi_i(g^N, c_k) = c_k\left(g^N\right) = 0$$

and

$$3\varphi_i(g^N, c_3) = c_3\left(g^N\right) = 1$$

■

Recall that $C^{(3)}$ is a subspace of functions whose value on a given network g, depends only on the number of links that form such network. The next Corollary characterizes the solutions on network games with these value functions in terms of linearity, symmetry and efficiency. It turns out that among all linear symmetric solutions, the egalitarian solution is characterized as the unique efficient solution on $C^{(3)}$. Formally,

Corollary 3 *Let $\varphi : G^{(3)} \times \Gamma^{(3)} \to \mathbb{R}^3$ be a linear, symmetric and efficient solution. Then for all $\omega \in C^{(3)}$:*

$$\varphi_i(g^N, \omega) = \frac{\omega(g^N)}{3}$$

In other words, all linear, symmetric and efficient solutions (e.g. Myerson's value) coincide with the egalitarian solution when restricted to these type of games, $C^{(3)}$

Now, another immediate application is to provide a characterization of all linear, symmetric and efficient solutions.

Theorem 3 *The solution $\varphi : G^{(3)} \times \Gamma^{(3)} \to \mathbb{R}^3$ satisfies linearity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(g^N, \omega) = \frac{\omega\left(g^N\right)}{3} + \sum_{k=1}^{2}\beta_k \left[\sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)\neq k}} (3-k)\omega(g)\right] \tag{15}$$

for some real numbers $\{\beta_1, \beta_2\}$.

Proof. *Let $\varphi : G^{(3)} \times \Gamma^{(3)} \to \mathbb{R}^3$ be a linear, symmetric and efficient solution; and $\omega \in \Gamma^{(3)}$. Then, by Proposition 3, Schur's Lemma and Proposition 6:*

$$\begin{aligned}
\varphi_i(g^N, \omega) &= \sum_{k=1}^{3} a_k \varphi_i\left(g^N, c_k\right) + \sum_{k=1}^{2} \varphi_i\left(g^N, z_k^k\right) \\
&= a_3 \varphi_i\left(g^N, c_3\right) + \sum_{k=1}^{2} \beta_k \left(z_k\right)_i \\
&= \frac{\omega\left(g^N\right)}{3} + \sum_{k=1}^{2}\beta_k \left[\sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)\neq k}} (3-k)\omega(g)\right]
\end{aligned}$$

■

Corollary 4 *The space of all linear, symmetric and efficient solutions on $G^{(3)} \times \Gamma^{(3)}$ has dimension $|\{\beta_1, \beta_2\}| = 2$.*

Example 2 *The Myerson value ψ^M is a solution that satisfies the axioms of linearity, symmetry and efficiency. Thus (for $n = 3$), ψ^M is of the form (15) and its corresponding parameters are $\beta_1 = 1/6$ and $\beta_2 = 0$.*

5 Conclusion

The point of view that we take in this article depends heavily on a decomposition of the space of value functions as a direct sum of 'special' subspaces. In the cases when $n = 3, 4$, it was decomposed as a direct sum of three orthogonal subspaces: a subspace of anonymous value functions, another subspace which we call $R^{(n)}$ and a subspace $T^{(n)}$ (wich is zero for the case of $n = 3$ nodes) that plays only the role of the common kernel of every linear symmetric solution. Although $R^{(n)}$ does not have a natural definition in terms of well known network theoretic considerations, it has a simple characterization in terms of vectors all of whose entries add up to zero.

Characterizations of solutions follow from such decomposition in an very economical way. So, an open challenge is to obtain the general decomposition for $\Gamma^{(n)}$ into direct sum of irreducible subspaces; since mathematically, the general case seems to has a much more complicated structure.

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