ON QUASI DROP PROPERTY OF UNBOUNDED SETS IN LOCALLY COMPLETE SPACES

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Abstract. In locally complete spaces with the sMc every unbounded closed convex set C with the quasi drop property has the Mackey (α)-property. In the frame of reflexive acyclic (LF)-spaces a quasi converse is obtained.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and B_X its closed unit ball. By the drop $D(x, B_X)$ defined by an element $x \in X \setminus B_X$ we mean the set $conv(\{x\} \cup B_X)$. Danes [2] proved that, for any Banach space $(X, \|\cdot\|)$ and every non-empty closed set $A \subset X$ at positive distance from B_X, there exists an $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$. Motivated by Danes theorem, Rolewicz [26] introduced the notion of drop property for the norm of a Banach space: the norm $\|\cdot\|$ in X has the drop property if for every non-empty closed set A disjoint from B_X there exists $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$. He proved that if the norm has the drop property then $(X, \|\cdot\|)$ is reflexive (see [26] Theorem 5). Later, Montesinos (see [18] Theorem 4) proved that a Banach space is reflexive if and only if it can be renormed to have the drop property.

Let B be a subset of a Banach space $(X, \|\cdot\|)$. The Kuratowski index of noncompactness of B, $\alpha(B)$, is the infimum of all positive numbers r such that B can be covered by a finite number of sets of diameter less than r. Given $f \in X^*$ such that $\|f\| = 1$ and $0 < \delta \leq 2$, consider the slice $S(f, B_X, \delta) = \{x \in B_X : f(x) \geq 1 - \delta\}$. The norm $\|\cdot\|$ in a Banach space X has property (α), if $\lim_{\delta \to 0} \alpha(S(f, B_X, \delta)) = 0$ for every $f \in X^*$, with $\|f\| = 1$. Also, Rolewicz ([26] Theorem 4), proved that if the norm has the drop property then it has property (α), and Montesinos ([18] Theorem 3) established that these two properties are equivalent.

Giles, Sims and Yorke [7] said that the norm has the weak drop property if for every non-empty weakly sequentially closed set A disjoint from B_X, there exists an $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$, and they proved that this property is equivalent to $(X, \|\cdot\|)$ being reflexive. Kutzarova [12] and Giles and Kutzarova [6] extended the discussion of these drop properties to closed bounded convex sets in Banach spaces. Cheng, Zhou and Zang [1], Zheng [30] and other authors studied those drop properties in locally convex spaces: a bounded, convex and closed subset B of a locally convex space (E, τ) is said to have the drop property if it is non-empty and for every non-empty sequentially closed subset $A \subset E$ disjoint from B there exists $a \in A$ such that $D(a, B) \cap A = \{a\}$.

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Qiu in [22] and Monterde and Montesinos in [16], introduced another drop properties in locally convex spaces: a non-empty closed convex bounded subset B of a locally convex space (E, τ) is said to have the quasi weak drop (resp. quasi drop) property if for every non-empty weakly closed (resp. closed) subset $A \subset E$ disjoint from B, there exists an $x_0 \in A$ such that $D(x_0, B) \cap A = \{x_0\}$. In [22] and [23], Qiu established a number of equivalences for the quasi weak drop property in Frèchet spaces and in quasi-complete locally convex spaces. He characterized reflexivity of those spaces by the condition that every closed bounded convex subset of the space must satisfy the quasi weak drop property. Concerning drop properties and their applications, see for example [1]-[7], [12]-[19], [21]-[26], [29] and [30]. In [15] can be found a extensive compilation of extensions and equivalent variational principles to drop properties.

In [13], Kutzarova and Rolewicz have dropped the boundedness assumption for the drop property in Banach spaces and proved

Theorem 1. *Let C be an unbounded closed convex set in a reflexive Banach space. The following conditions are equivalent:*
i) C has the drop property;
ii) $int(C) \neq \emptyset$ and C has the property (α)

They asked if the existence of such a closed convex unbounded set C with the drop property forces the space to be reflexive. In [19], Montesinos proved that this is the case. Later, in [14], Lin and Yu proved that if C is an unbounded closed convex set with the weak drop property in a Banach space, then C has nonempty interior and the Banach spece is reflexive.

In [4], the author considered locally convex spaces with the strict Mackey convergence condition (sMc, see below) and studied the relation between the quasi drop property and the defined *Mackey (α)-property* (see below). Then he characterized quasi drop property for bounded disks in Frechet spaces. Later, in [5] on the base of techniques from Kutzarova-Rolewicz [13], Montesinos [19] and Lin-Yu [14] the Kutzarova-Rolewicz's Theorem was extended to the family of reflexive Fréchet spaces, i.e. in a reflexive Fréchet space an unbounded closed convex subset C has the quasi drop property if and only if $int(C) \neq \emptyset$ and C has the Mackey (α)-property.

Recently, Monterde and Montesinos [17] proved that if a closed convex bounded subset B of a locally convex space has the quasi drop property then it has property (α). And if the locally convex space is complete, property (α) implies weak compactness.

Based again on results and techniques of [13],[14] and [19], in this note is proved that in a locally complete locally convex space (E, τ) with the sMc every closed convex unbounded subset $B \subset E$ satisfying the quasi drop property has the Mackey (α)-property. Also it is proved that subsets of acyclic (LF)-spaces $(E, \tau) = ind(E_n, \tau_n)$ have both quasi drop, Mackey (α)-properties and non empty interior if there are certain subsets satisfying them in every step (E_n, τ_n). Wich is a kind of converse to the previous result or an extension to Kutzarova-Rolewicz theorem.

For the sake of completeness some results appeared in [5] are included in this work.

2. PRELIMINARIES

A closed, bounded and absolutely convex subset is called a disk. If D is a disk in the locally convex space (E, τ) then we let E_D denotes the linear span of D, equipped with the topology given by ρ_D the gauge (Minkowski's functional) of D. This topology has a base of zero neighborhoods of the form $\{aD : a > 0\}$, and makes E_D into a normed space such that $\tau|_{E_D} \leq \rho_D|_{E_D}$, for τ the original topology of E. And (E, τ) is said to be locally complete if every disk $D \subset E$, is a Banach disk, that is (E_D, ρ_D) is a Banach space. Note that for metrizable spaces, completeness and local completeness are equivalent. For local completeness, see [10] and [20].

According to Grothendieck (see [9]), we have that a space (E, τ) satisfies the strict Mackey condition (sMc) if for every bounded subset $B \subset (E, \tau)$, there exists a disk $D \subset E$ containing B such that the topologies of E and E_D agree on B, i.e. $\tau|_B = \rho_D|_B$ and so on every subset of B. Note that every metrizable, unordered Baire like or strictly barreled space satisfies the sMc (see [27]; [20], 5.1.27(ii) and [8], Corollary 3.4). So, in particular, every Fréchet or acyclic (LF)-space is locally complete and satisfies the sMc.

For a τ-closed convex set $C \subset E$, denote by $F(C)$ the set of all τ-continuous linear functionals $f \in (E, \tau)' \setminus \{0\}$ which are bounded above on C. For $f \in F(C)$ let $M_f = \sup\{f(x) : x \in C\}$, and for $\delta > 0$ consider the slice $S(f, C, \delta) = \{x \in C : f(x) \geq M_f - \delta\}$. The set C is said to have the (α)-property with respect to τ if for every $f \in F(C)$ and for every neighborhood U of 0 in τ, there exists $\delta > 0$ such that $S(f, C, \delta)$ can be covered by a finite number of translates of U.

If a slice $S(f, C, \delta_0)$ is bounded in a locally complete locally convex space (E, τ) which satisfies the sMc, then there exists a Banach disk $D \subset E$ containing $S(f, C, \delta_0)$ such that $\tau\big|_{S(f,C,\delta_0)} = \rho_D\big|_{S(f,C,\delta_0)}$. In this case, the Kuratowski index of non-compactness of $S(f, C, \delta_0)$ associated to the disk D is $\alpha_D(S(f, C, \delta_0))$ the infimum of all positive numbers r such that $S(f, C, \delta_0)$ is covered by a finite number of sets of ρ_D-diameter less than r. The τ-closed convex set $C \subset E$ is said to have the Mackey (α)-property if for every $f \in F(C)$ and D as above $\lim_{\delta \to 0} \alpha_D(S(f, C, \delta)) = 0$. In this case, due to the fact that ρ_D and τ induce the same topology on the slice, we get that C has the (α)-property with respect to τ. Obviously, if $(E, \|\cdot\|)$ is a normed space both (α)-properties coincide.

3. RESULTS

Proposition 1. *Let C be a non-empty closed convex (unbounded) subset of the locally convex space (E, τ). Suppose that C has the quasi drop property. Then every C-stream in E has a τ-convergent subsequence.*

Proof. Suppose there exists a sequence $(x_n)_n \in E$ such that $x_{n+1} \in D(x_n, C) \setminus C$, for every $n \in \mathbb{N}$, but $(x_n)_n$ does not have any τ-convergent subsequence. So, for every subsequence $(x_{n_k})_k \subset (x_n)_n$ we have that $A = \{x_{n_k} : k \in \mathbb{N}\}$ is a closed set and C does not have the quasi drop property. □

Proposition 2. *Let C be a non-empty closed convex unbounded subset of the locally convex space (E, τ). Let $f \in F(C)$ and $M_f := \sup\{f(x) : x \in C\}$. Suppose that C has the quasi drop property. Then for every $\delta > 0$, the slice $S(f, C, \delta) = \{x \in C : f(x) \geq M_f - \delta\}$ is a bounded set.*

Proof. Suppose this is not true. Then there exist $f_0 \in F(C)$, $\delta_0 > 0$ and $U_0 \in \tau$ an open convex and simetric zero neighborhood such that for every $R > 0$ we have that $S(f_0, C, \delta_0)$ is not contained in RU_0; or equivalently, for every $R > 0$ there exists $x_R \in S(f_0, C, \delta_0)$ such that $\rho_{U_0}(x_R) > R$, where $\rho_{U_0}(\cdot)$ is the τ-continuous Minkowski's seminorm generated by U_0.

Let $M \geq M_{f_0}$. Let $x_1 \in E$ be such that $f_0(x_1) > M_{f_0}$. Find $0 < \lambda_1 < 1$ such that $(1-\lambda_1)f_0(x_1) - \lambda_1 M > M_{f_0}$.

Take $\overline{x_2} \in S(f_0, C, \delta_0)$ satisfying $\rho_{U_0}(\lambda_1\overline{x_2} + (1-\lambda_1)x_1 - x_1) \geq 1$. Let $x_2 := \lambda_1\overline{x_2} + (1-\lambda_1)x_1$. So, $\rho_{U_0}(x_2 - x_1) \geq 1$ and

$$\begin{aligned} f_0(x_2) &= \lambda_1 f_0(\overline{x_2}) + (1-\lambda_1)f_0(x_1) \geq (1-\lambda_1)f_0(x_1) - \lambda_1 |f_0(\overline{x_2})| \\ &\geq (1-\lambda_1)f_0(x_1) - \lambda_1 M > M_{f_0} \end{aligned}$$

Suppose now, we have found $x_1, x_2, ..., x_n \in E$ such that $x_{i+1} \in D(x_i, C) \setminus C$, for $i = 1, 2, ..., (n-1)$, with $f_0(x_i) > M_{f_0}$, for $i = 1, 2, ..., n$ and $\rho_{U_0}(x_i - x_j) \geq 1$ for $i, j = 1, 2, ..., n$ and $i \neq j$. In order to find x_{n+1}, find $0 < \lambda_n < 1$ such that $(1-\lambda_n)f(x_n) - \lambda_n M > M_{f_0}$ and take $\overline{x_{n+1}} \in S(f_0, C, \delta_0)$ satisfying $\rho_{U_0}(\lambda_n\overline{x_{n+1}} + (1-\lambda_n)x_n - x_i) \geq 1$, for all $i = 1, 2, ..., n$. Let $x_{n+1} := \lambda_n\overline{x_{n+1}} + (1-\lambda_n)x_n$. So, $\rho_{U_0}(x_{n+1} - x_i) \geq 1$, for all $i = 1, 2, ..., n$; and

$$\begin{aligned} f_0(x_{n+1}) &= \lambda_n f_0(\overline{x_{n+1}}) + (1-\lambda_n)f_0(x_n) \geq (1-\lambda_n)f_0(x_n) - \lambda_n |f_0(\overline{x_{n+1}})| \\ &\geq (1-\lambda_n)f_0(x_n) - \lambda_n M > M_{f_0} \end{aligned}$$

Then the sequence $(x_n)_n$ is a C-stream in E with no convergent subsequences. This is a contradiction. □

Theorem 2. *Let (E, τ) be a locally complete locally convex space with the strict Mackey condition (sMc) and $C \subset E$ be a closed convex unbounded subset. Suppose that C has the quasi-drop property. Then C has the Mackey (α)-property.*

Proof. Let $f \in F(C)$. Find $x_0 \in E$ such that $f(x_0) > M_f$. We may assume that $M_f > 1$, then by proposition 2, the slice $S_1 := S(f, C, 1)$ is a τ-bounded closed set and $B := cvx\{S_1 \cup \{x_0\}\}$ is a bounded closed convex subset of E. Since (E, τ) is locally complete and has the sMc, there exists $D \subset E$ a Banach disk such that $B \subset D$ and $\tau|_B = \rho_D|_B$. In particular, $\tau|_{S_1} = \rho_D|_{S_1}$. If $\inf\{\alpha_D(S(f, C, \varepsilon) : 1 > \varepsilon > 0\} > 2\delta_0$, for some $\delta_0 > 0$, then (see [26], Theorem 4) for every finite dimensional subspace $L \subset E_D$ we have:

$$\sup_{x \in S(f,C,\varepsilon)} (\inf_{y \in L} \rho_D(x-y)) \geq \frac{1}{2} \inf_{\varepsilon > 0} \alpha_D(S(f, C, \varepsilon)) > \delta_0 \qquad \cdots (1)$$

Take $\varepsilon_1 < f(x_0) - M_f$. And choose $\overline{x_1} \in S(f, C, \varepsilon_1)$ such that

$$\inf\{\rho_D(\overline{x_1} - z) : z \in span\{x_0\}\} > \delta_0.$$

Let $x_1 = \frac{x_0 + \overline{x_1}}{2}$, then

$$f(x_1) = f(\frac{x_0 + \overline{x_1}}{2}) = \frac{f(x_0)}{2} + \frac{f(\overline{x_1})}{2} > \frac{M_f + \varepsilon_1}{2} + \frac{M_f - \varepsilon_1}{2} = M_f.$$

Moreover

$$\inf\{\rho_D(x_1 - z) : z \in span\{x_0\}\} = \frac{1}{2}\inf\{\rho_D(\overline{x_1} - z) : z \in span\{x_0\}\} > \frac{\delta_0}{2}$$

Now, suppose we have $\{x_0, x_1, ..., x_n\}$, such that $x_i \neq x_j$ if $i \neq j \leq n$, and
i) $f(x_i) > M_f$

ii) $\inf\{\rho_D(x_i - z) : z \in span\{x_0, ..., x_{i-1}\}\} > \frac{\delta_0}{2}$
iii) $x_i \in D(x_{i-1}, C)$
for every $i \leq n$. Take $\varepsilon_{n+1} < f(x_n) - M_f$ and by (1) find $\overline{x_{n+1}} \in S(f, C, \varepsilon_{n+1})$ such that

$$\inf\{\rho_D(\overline{x_{n+1}} - z) : z \in span\{x_0, x_1, ..., x_n\}\} > \delta_0.$$

Let $x_{n+1} = \frac{x_n + \overline{x_{n+1}}}{2}$ then, in an analogous way to x_1, $f(x_{n+1}) > M_f$ and

$$\begin{aligned} &\inf\{\rho_D(x_{n+1} - z) : z \in span\{x_0, ..., x_n\}\} \\ = \ &\frac{1}{2}\inf\{\rho_D(\overline{x_{n+1}} - z) : z \in span\{x_0, ..., x_n\}\} > \frac{\delta_0}{2}. \end{aligned}$$

Then the sequence $(x_n)_n$ satisfies (i,ii,iii) and the set $A = \{x_0, x_1, ..., x_n, ...\} \subset B$ is ρ_D-closed. Since the topologies τ and ρ_D agree on B, A is τ-closed and $A \cap C = \emptyset$. Hence C, does not have the quasi drop property. This is a contradiction. □

Recall a generalization of Cantor's intersection theorem due to Kuratowski [11]

Lemma 1. *Given a complete metric space and a sequence of non-empty closed sets $\{F_n\}_{n\in\mathbb{N}}$, $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ with the property that, for α the Kuratowski index of noncompactness, $\lim_n \alpha(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n$ is non-empty and compact.*

Remark 1. *a) Suppose that C has the Mackey (α)-property in the locally complete locally convex space (E, τ) which has the sMc, let $f \in F(C)$ and a sequence $(\varepsilon_n)_n \in \mathbb{R}^+$ convergent to zero. Then for every sequence $(x_n)_n \in C$ such that $x_n \in S(f, C, \varepsilon_n)$ there is a subsequence $(x_{n_k})_k \subset (x_n)_n$ convergent in C.*

b) In particular, under the assumptions of Theorem 2, every $f \in F(C)$ attains its supremum on C.

Proposition 3. *Let C be a closed convex unbounded subset of the Fréchet space (E, τ). Suppose that C has the quasi-drop property. Then $int(C) \neq \emptyset$.*

Proof. Since C is not bounded, there exists $(x_n)_n \in C$ such that does not have any convergent subsequence. For every $x \notin C$, define $y_n := \frac{1}{2^n}x + \sum_{i=1}^{n} \frac{1}{2^{n-i+1}}x_i$. So, $(y_n)_n$ is a sequence which has non-empty intersection with C. If this is not true, then we have two possibilities:

i) There exists a subsequence $(y_{n_k})_k \subset (y_n)_n$ which does not have convergent subsequences. Then $A := \{y_{n_k} : k \in \mathbb{N}\}$ is closed disjoint to C, and contradicts the quasi drop property of C.

ii) Every subsequence $(y_{n_k})_k \subset (y_n)_n$ has a convergent subsequence. Let $(y_{n_k})_k \subset (y_n)_n$ be convergent to $y_0 \in E$. This implies that $y_{n_k+1} \to y_0$, too. Since $y_{n_{k+1}} = \frac{1}{2}(y_{n_k} + x_{n_k+1})$ and $x_{n_k+1} = 2y_{n_k+1} - y_{n_k}$, then $(x_{n_k+1})_k$ converges to y_0. This is not possible.

Hence, $\{y_n : n \in \mathbb{N}\} \cap C \neq \emptyset$. Now, given $z \in E$ and $L \neq 0$, define the homeomorfism $T_{z,L} : E \to E$ given by $T_{z,L}(x) = z + L(x - z)$, and the application $T_{(x_1,...,x_n)}(x) = T_{x_1,2} \circ T_{x_2,2} \circ \cdots \circ T_{x_n,2}(x)$. It is easy to verify that for the elements $\{x_n : n \in \mathbb{N}\}$ of the original sequence we have that $y_n \in C$ if and only if $x \in T_{(x_1,x_2,...,x_n)}(C)$. Then $E \setminus C = \bigcup_{n\in\mathbb{N}} T_{(x_1,x_2,...,x_n)}(C)$. Since (E, τ) is a Fréchet space, the Baire category theorem implies that $int_\tau(C) \neq \emptyset$. □

Proposition 4. *Let (E,τ) be a Frèchet space and $C\subset E$ be a closed convex unbounded subset with $int_\tau(C)\neq\emptyset$. Suppose that C has the Mackey (α)-property. Then for every $b\in E\setminus C$ the set $D(b,C)$ is τ-sequentially closed. Equivalently, $D(b,C)$ is τ-closed.*

Proof. Suppose there is a point $b\in E\setminus C$ such that $D(b,C)$ is not τ-closed. Then there exists $a\in\overline{D(b,C)}^{\tau}$ such that $a\notin D(b,C)$. So, there are sequences $(y_n)_n\in C$ and $(\lambda_n)_n\in[0,1]$ such that $x_n:=\lambda_n b+(1-\lambda_n)y_n\to a$ respect to τ. Then the sequence $\lambda_n\to 1$ and for every τ-continuous seminorm ρ such that distinguish a subsequence $(y_{n_k})_k\subset(y_n)_n$ we have that $\rho(y_{n_k})\to\infty$. By a convexity argument the ray $r=\{b+\eta(a-b):\eta\geq 1\}$ is contained in $\overline{D(b,C)}^{\tau}\setminus D(b,C)$. Note that $r\cap C=\emptyset$ and $int_\tau(C)\neq\emptyset$ imply that there exists $f\in(E,\tau)'\setminus\{0\}$ such that $M_f:=\sup\{f(x):x\in C\}\leq\inf\{f(x):x\in r\}$, evenmore $f(a)=f(b)$, so $r\subset H=\{x\in E:f(x)=f(a)\}$. Then $M_f\leq f(a)$. Since C has the Mackey (α)-property, for every $\delta>0$ the set $S(f,C,\delta)$ is bounded. Consider the set $A:=\{a,b\}\cup\{x_n:n\in\mathbb{N}\}\cup S(f,C,\delta)$. Since A is bounded and (E,τ) has the sMc, there exists a Banach disk $B\subset E$ such that $A\subset B$ and $\rho_B\,|_A=\tau\,|_A$, evenmore if we make $C_B:=C\cap E_B$ then $\{y_n:n\in\mathbb{N}\}\subset C_B\subset E_B$ and $x_n\to a$ respect to ρ_B, so $\rho_B(y_n)\to\infty$. Then we have that $a\in\overline{D(b,C_B)}^{\rho_B}$ but $a\notin D(b,C_B)$. Note also that $int_{\rho_B}(C_B)\neq\emptyset$ and $f_B:=f\,|_{E_B}\in(E_B,\rho_B)'\setminus\{0\}$, so $f_B\in F(C_B)$ and f_B separates r and C_B. Hence all the previous construction and observations remains valid in the Banach space (E_B,ρ_B). If we prove that $a\in D(b,C_B)$, which clearly is contained in $D(b,C)$ we are done. But in these conditions the proof continues exactly as the rest of proof at this point of Proposition 5 in [13], where ρ_B substitutes $\|\cdot\|$. □

Note that Proposition 1 in [13] has been proved above for Fréchet spaces. Also, Lemma 2 and Lemma 12 in [13] follow directly being true in the frame of reflexive Fréchet spaces. Then Remark 2(iii) in [14] can be extended to

Remark 2. *Let (E,τ) be a reflexive Fréchet space and $C\subset E$ an unbounded closed convex subset. Suppose that C has the Mackey (α)-property and that $int(C)\neq\emptyset$ then C contains a ray $\{c+\lambda b:\lambda\geq 0\}$. Moreover, for any $x\in E$ there is $\beta>0$ such that C contains the ray $\{x+\lambda b:\lambda\geq\beta\}$*

Theorem 3. *Let (E,τ) be a reflexive Fréchet space and $C\subset E$ be an unbounded closed convex subset. Then the following conditions are equivalent:*
a) C has the quasi-drop property
b) $int(C)\neq\emptyset$ and C has the Mackey (α)-property.

Proof. Assume that C does not have the quasi drop property. So, there is a closed set $A\subset E$ disjoint to C such that for every $x\in A$ there is $a\in A\setminus\{x\}$ satisfying $a\in A\cap D(x,C)$. Take any point $x_1\in A$. Put $d_1':=\inf\{d(a,C):a\in A\cap D(x_1,C)\}$ and find $x_2\in A\cap D(x_1,C)$ such that $d_2:=d(x_2,C)<d_1'+1$. Choose $\{x_1,x_2,...,x_n\}$ such that $x_{i+1}\in A\cap D(x_i,C)$ and $x_{i+1}\neq x_i$ for $i=1,...,n-1$ and if we make $d_i':=\inf\{d(a,C):a\in A\cap D(x_i,C)\}$ then $d_{i+1}:=d(x_{i+1},C)<d_i'+\frac{1}{i}$. Inductively construct, in this way, a C-stream $\{x_n:n\in\mathbb{N}\}$ with these characteristics, and note that $(d_n)_n\subset\mathbb{R}$ is a convergent sequence to some $\varepsilon_0\geq 0$. Note that this C-stream $\{x_n:n\in\mathbb{N}\}$ does not have any convergent subsequence. In order to see this, suppose that the C-stream possess convergent subsequences and consider two cases:

a) $\varepsilon_0 = 0$. This means that there is a sequence $(y_n)_n \in C$ such that $d(x_n, y_n) \to 0$. Let $A_1 := cvx\{x_n : n \in \mathbb{N}\}$, by the lemma in [18], $A_1 \cap C = \emptyset$ and since $int(C) \neq \emptyset$, there exists $f \in (E,\tau)' \setminus \{0\}$ which separates A_1 and C. We may assume that $f \in F(C)$. For $M_f := \sup\{f(x) : x \in C\}$, we have that $f(y_n) \to M_f$. By the Mackey (α)-property, the Kuratwoski's lemma guarantees the existence of a subsequence $(y_{n_k})_k \subset (y_n)_{\mathbb{N}}$ which is convergent to some $y_0 \in C$ and $f(y_0) = M_f$. Then $d(x_n, y_o) \to 0$ and since A is closed we get that $y_0 \in A \cap C$. This is a contradiction.

b) If $\varepsilon_0 > 0$ and $(x_n)_{\mathbb{N}}$ has a convergent subsequence to some $x_0 \in A \cap \bigcap_{i\in\mathbb{N}} D(x_i, C)$, i.e. $x_0 \in A \cap D(x_i, C)$, for every $i \in \mathbb{N}$. Then there exists $a \in A \setminus \{x_0\}$ satisfying $a \in A \cap D(x_0, C)$ and $d(a, C) < d(x_0, C)$. Find $n \in \mathbb{N}$ such that $\frac{1}{n} < d(x_0, C) - d(a, C)$, then $d(x_{n+1}, C) > d(x_0, C) > d(a, C) + \frac{1}{n} \geq d'_n + \frac{1}{n}$. Which is a contradiction. Then the C-stream does not have any convergent subsequence.

Now, by the Remark 2, there exists $b \in E \setminus \{0\}$ such that for every $x \in E$ there is $\beta > 0$ such that C contains the ray $\{x + \lambda b : \lambda \geq \beta\}$.

Let $\eta := \sup\{\beta : (\beta b + \{x_n\}_{\mathbb{N}}) \cap C = \emptyset\}$. Note that

i) $\eta b + C \subset C$

ii) if $\eta b + x_n \in C$, then $\eta b + x_m \in C$ for every $m > n$.

So, for every convex combination $\sum_{i=1}^{n} a_i x_i$ where each $a_i \geq 0$ and $\sum_{i=1}^{n} a_i = 1$, if $\eta b + \sum_{i=1}^{n} a_i x_i \in int(C)$ then

$$\eta b + x_{n+1} \in cvx\left\{\left(\eta b + \sum_{i=1}^{n} a_i x_i\right) \cup (\eta b + C)\right\} \subset int(C).$$

Which is not possible. Then $(\eta b + cvx\{x_n : n \in \mathbb{N}\}) \cap int(C) = \emptyset$ and there exists $f \in (E,\tau)' \setminus \{0\}$ such that

$$\inf\{f(\eta b + x_n) : n \in \mathbb{N}\} = M_f = \sup\{f(x) : x \in C\}$$

By the definition of η, there exists a sequence $(y_{n_k})_k \in C$ such that

$$d(\eta b + x_{n_k}, y_{n_k}) \to 0 \text{ and } f(y_{n_k}) \to M_f$$

Since C has the Mackey (α)-property there exists a subsequence $(y_{n_l})_l \subset (y_{n_k})_k$ which is convergent to some $y_0 \in C$. Then the sequence $(x_n)_n$ has a convergent subsequence. This is a contradiction. □

Recall an (LF)-space $(E,\tau) = ind(E_n, \tau_n)$ is acyclic if and only if for every one of the Frèchet spaces (E_n, τ_n) there is an absolutely convex 0-neighbourhood U_n with

1. $U_n \subset U_{n+1}$, for every $n \in \mathbb{N}$
2. for every $n \in \mathbb{N}$ there is $m > n$ such that all topologies of the Frèchet spaces (E_k, τ_k) for $k > m$ coincides on U_n.

Equivalently (see [28]), the (LF)-space $(E,\tau) = ind(E_n, \tau_n)$ is acyclic if and only if it is boundedly retractive, that is, for every bounded subset $B \subset E$ there is $n \in \mathbb{N}$ such that B is contained in E_n and the topologies τ and τ_n coincide on B. Recall acyclic (LF)-spaces appear frecuently in applications on distributions theory.

Remark 3. *Let $(E,\tau) = ind(E_n, \tau_n)$ be an acyclic (LF)-space, where every (E_n, τ_n) is a reflexive Frèchet space, so (E,τ) is also a reflexive space (see* [10]*, Proposition 11.4.5). Suppose that for every $n \in \mathbb{N}$, there is an unbounded and τ_n-closed convex subset $C_n \subset E_n$ such that $C_n \subset C_{n+1}$ and for every $n \in \mathbb{N}$ there exists $k > n$ such that $C \cap E_n \subset C_k$. Suppose also, that every C_n has the quasi drop property on the space (E_n, τ_n), equivalently (by the previous Theorem), suppose every C_n has the Mackey (α)-property and $int_{\tau_n}(C_n) \neq \emptyset$. Then $C = \bigcup\limits_{n\in\mathbb{N}} C_n \subset E$ is such that*

i) $C \cap E_n$ has non empty τ_n-interior, for every $n \in \mathbb{N}$, since $C_n \subset C \cap E_n$. Then $int_\tau(C) \neq \emptyset$ and in an analogous way C is an τ-closed convex unbounded subset of (E,τ).

ii) Let $f \in (E,\tau)'$ be such that $f \in F(C)$ and $(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence of positive numbers such that $\varepsilon_i \to 0$. By Proposition 2, the slice $S(f,C,\varepsilon_1)$ is bounded in (E,τ), which is boundedly retractive. So, there exists $n_0 \in \mathbb{N}$ such that $S(f,C,\varepsilon_1)$ is contained and bounded in (E_{n_0}, τ_{n_0}), even more $\tau\left|_{S(f,C,\varepsilon)} = \tau_{n_0}\right|_{S(f,C,\varepsilon)}$ Note $f|_{En}$ is τ_n-continuous and $f \in F(C_n)$ for every $n \in \mathbb{N}$. On the other hand, there exists $k > n_0$ such that $S(f,C,\varepsilon_1) \subset C \cap E_{n_0} \subset C_k$. Since C_k has the Mackey (α)-property, there exists $D \subset E_k$ a Banach disk such that $\alpha_D(S(f,C_k,\varepsilon_i)) \to 0$ if $i \to \infty$. But $S(f,C,\varepsilon_i) \subset S(f,C_k,\varepsilon_i)$, then $\alpha_D(S(f,C,\varepsilon_i)) \to 0$ if $i \to \infty$. And C has the Mackey (α)-property.

iii) Under these assumptions on the (LF)-space (E,τ) and the subset $C \subset E$, for every $b \in E \setminus C$ the set $D(b,C)$ is τ-sequentially closed. It follows from proposition 4 and the sequential retractivity of the acyclic (LF)-space (E,τ).

In order to see that C has the quasi drop property, based on the same hypothesis and the previous observations (i,ii,iii), suppose that C does not have the quasi drop property. Then there exists a τ-closed subset $A \subset E$ such that for every $x \in A$ there is $a \in A \setminus \{x\}$ such that $a \in A \cap D(x,C)$. So, construct a stream in the following way, take any $x_1 \in A_1 := A \cap E_1$, then there exists $a_1 \in A_{n_1'} \setminus \{x_1\}$ for some n_1' such that $a_1 \in A_{n_1'} \cap D(x_1,C) := (A \cap E_{n_1'}) \cap D(x_1,C)$. But $D(x_1,C) \cap E_{n_1'} \subset D(x_1, C_{n_1})$ for some $n_1 > n_1'$ then $a_1 \neq x_1$ is such that $a_1 \in A_{n_1} \cap D(x_1, C_{n_1})$. Since C_{n_1} has the quasi drop property in (E_{n_1}, τ_{n_1}), there exists $x_2 \in A_{n_1} = A \cap E_{n_1}$ such that $\{x_2\} = \{A_{n_1} \cap D(x_1,C)\} \cap D(x_2, C_{n_1}) \subset A \cap D(x_1,C)$.

In this way, form a stream $\{x_n : n \in \mathbb{N}\} \subset A$ such that there exists a sequence of natural numbers $(n_i)_{i\in\mathbb{N}}$ with $x_{i+1} \in A_{n_i} := A \cap E_{n_{i\times}}$, another sequence $(a_i)_{i\in\mathbb{N}} \in A$ such that for every $i \in \mathbb{N}$, $a_i \neq x_i$ and $a_i \in A_{n_i} \cap D(x_i, C_{n_i})$. Even more, $\{x_{i+1}\} = (A_{n_i} \cap D(x_i,C)) \cap D(x_{i+1}, C_{n_i})$.

Theorem 4. *Let $(E,\tau) = ind(E_n,\tau_n)$ be a reflexive acyclic (LF)-space and $C = \bigcup\limits_{n\in\mathbb{N}} C_n$ both as in remark 3. Then C satifies quasi drop property, Mackey (α)-property and $int_\tau C \neq \emptyset$.*

Proof. On the basis of the proof of previous theorem 3. For every $n \in \mathbb{N}$, consider a metric d_n on the space E_n which defines the topology τ_n. Suppose that C does not have the quasi drop property and construct a C-stream in the following way, take again a τ-closed set A such that for every $x \in A$ there is $a \in A \setminus \{x\}$ such that $a \in A \cap D(x,C)$. Take any point $x_1 \in A$, we may assume $x_1 \in A_1 = A \cap E_1$. Let $\widetilde{\rho_1} := \inf\{d_1(a,C) : a \in A_1 \cap D(x_1,C)\}$. Then find $x_2 \in A_2 \cap D(x_1,C)$ such that $\rho_2 := d_2(x_2,C) < \widetilde{\rho_1} + 1$. Then, let $\widetilde{\rho_2} := \inf\{d_2(a,C) : a \in A_2 \cap D(x_2,C)\}$. Find $x_3 \in A_3 \cap D(x_2,C)$ such that $\rho_3 := d_3(x_3,C) < \left\{\widetilde{\rho_1} + \frac{1}{2}, \widetilde{\rho_2} + \frac{1}{2}\right\}$. Follow in this way to obtain $\{x_1, x_2, x_3, \cdots, x_n\}$ such that $x_{i+1} \in A_{i+1} \cap D(x_i, C)$ with $x_{i+1} \neq x_i$ for every $i = 1, \cdots, n-1$, and $\widetilde{\rho}_i := \inf\{d_i(a, C) : a \in A_i \cap D(x_i, C)\}$. Then, find $x_{n+1} \in A_{n+1} \cap D(x_n, C)$ such that $\rho_{n+1} := d_{n+1}(x_{n+1}, C) < \left\{\widetilde{\rho}_i + \frac{1}{i}\right\}_{i=1}^{n}$. Note that this stream $(x_n)_{n\in\mathbb{N}}$ does not have convergent subsequences. If the stream has any convergent subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$, by the sequential retractivity of the (LF)-space (E, τ), there exists $n_0 \in \mathbb{N}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ is contained in E_{n_0} and it is τ_{n_0}-convergent. So, it is exactly the same situation of cases (a) or (b) in the previous theorem. Then it is not possible and the C-stream does not have any convergent subsequence.

By the previous theorem, for every $n \in \mathbb{N}$ the set C_n contains a ray and also by remark 2, C contains a ray, too.

Let $\eta := \sup\{\beta : (\beta b + \{x_n\}_{\mathbb{N}}) \cap C = \emptyset\}$. Note that

i) $\eta b + C \subset C$

ii) if $\eta b + x_n \in C$, then $\eta b + x_m \in C$ for every $m > n$.

Then in analogous way to the previous theorem $(\eta b + cvx\{x_n : n \in \mathbb{N}\}) \cap int(C) = \emptyset$ and there exists $f \in (E, \tau)' \setminus \{0\}$ such that

$$\inf\{f(\eta b + x_n) : n \in \mathbb{N}\} = M_f = \sup\{f(x) : x \in C\}$$

By the definition of η, there exists a sequence $(y_{n_k})_k \in C$ such that

$$(\eta b + x_{n_k}) - y_{n_k} \to 0 \text{ and } f(y_{n_k}) \to M_f$$

Since C has the Mackey (α)-property there exists a subsequence $(y_{n_l})_l \subset (y_{n_k})_k$ which is convergent to some $y_0 \in C$. Then the sequence $(x_n)_n$ has a convergent subsequence. This is a contradiction. □

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