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An extension of the parametrized version of the Theorem of Sard Smale

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Abstract

1 Introduction

Mathematical modeling of many porblemas typical of economic theory naturally lead us to work with vector spaces whose positive cone has empty interior. Often the consumption set of economic agents is presented precisely as one of these sets.

Identifying the set of equilibria Walrasian these economies, it is common to use the excess utility function $e : \Delta \times \Omega \to R^{n-1}$, where Δ is the simplex of dimension $n-1$ and $\Omega = (B_+)n$ i.e, the Cartesian product of n copies of the positive cone of a Banach space B. The set of zeros of this function correspond to the set of Walrasian equilibria, *via* an homeomorphism. Then to analyze the structure of $e^{-1}(0)$ has importance for economic theory. Note that the classical arguments of the mathematical analysis can not be applied to do this analysis because the interior of the domain of e is empty.

The aim of this work is to extend the classical analysis to functions defined in the interior of convex sets is not necessarily nonempty.

2 Admissible directions and derivatives

Let B be a Banach space, which positive cone B_+ has empty interior. We will say that $h \in B$ is α admissible direction for $x \in B_+$, if and only if there exist $\alpha > 0$ such that $x + \alpha \frac{h}{\|h\|} \in B_+$.

Note that since B_+ is a convex subset of B, then if y and x are points in B_+ then $h = (y - x)$ is α admissible for x. To see this, consider $z = \alpha' y + (1 - \alpha' x),\ 0 \leq \alpha' \leq 1$ then $z \in B_+$ and $z = x + \alpha'(y - x)\ \forall\ 0 \leq \alpha$ and finally consider $\alpha = \frac{\alpha'}{\|h\|}$.

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It follows that if h is α admissible, is β admissible for all $0 < \beta \leq \alpha$.

We will symbolize by $\mathcal{A}_\alpha(x)$ the set of the $\alpha-$admissible directions at x, note that

$$\mathcal{A}_\alpha(x) \subseteq \mathcal{A}_\beta(x)\ \alpha \geq \beta > 0.$$

Let $u : B_+ \to R$ be a real function defined in B_+. It will be said that u is star-differentiable at $x \in B^+$ if there exist the Gteaux derivative of u at x for all $h \in \mathcal{A}_\alpha$. That is u is star-differentiable at x if and only if there exists a map $L_x \in L(B_+, R)$ such that

$$\lim_{\alpha\to 0} \frac{u(x+\alpha h) - u(x)}{\alpha} = \frac{d}{d\alpha}|_{\alpha=0} u(x+\alpha h) = L_x h$$

for all $h \in \mathcal{A}_\alpha(x)$. Equivalently if

$$u(x+\alpha h) - u(x) = \alpha L_x h + o(\alpha h)$$

being $L_x \in B^*$, where by B^* we symbolized the topological dual of B. We symbolize $u'(x) = L_x$.

3 The star-topology

Let $x \in B_+$ we define a star-neighborhood of x by

$$V_x^*(\alpha) = \left\{ y \in B_+ : y = x + \beta \frac{h}{\|h\|}, \forall\, h \in \mathcal{A}_\alpha(x),\ where\ 0 \leq \beta \leq \alpha \right\}.$$

We will say that O is an star-open subset of B_+ if for each $x \in O$ there exist $V_x^*(\alpha) \subset O$. As it is easy to see this sets conform a base of a topology, the star-topology.

4 Star-manifolds

Roughly speaking a star-manifold is a Hausdorff subset Γ of a Banach space that locally behaves like B^+.

A parametrization of a set $V \subset B$ of class C^k is an C^k homeomorphism $\phi : V_0 \to R$ defined on a star-open subset $V_0 \subset B_+ \to V$.

Definition 1 Star Manifold *We say that that Γ is a C^k star-manifold if for every $p \in \Gamma$, there exists an open star-neighborhood of B_+ symbolized by $V_a^*(\alpha)$ and a C^k parametrization, $\phi : V_a(\alpha) \to V_p \cap \Gamma$ where $V_p \subset B$ is a neighborhood of p. The pair $(V_p \cup \Gamma, \phi)$ is a star chart in Γ^*. A C^k star-atlas is a collection of star charts $(V_{p_i} \cup M, \phi_i)$ $i \in I$, which satisfies the following properties:*

(i) The collection $V_{p_i} \cup M,\ i \in I$ *cover* Γ.

(ii) Any two charts are compatible.

(iii) $\phi_i^{-1}(V_{p_i} \cup M)$ *is an star-open subset of* B_+. *A star manifold* Γ *will be symbolized by* Γ^*.

Given a chart $(V_p, \phi),\ \phi^{-1} : (V_p) \to V_a^*(\alpha)$ is an homeomorphism, onto an star open set.

Definition 2 Star Submanifold *Let* Γ^* *a* C^k-*Banach manifold* $k \geq 0$. *A subset* S *of* M^* *is called a star submanifold of* Γ^* *if and only if for each point* $x \in S$ *there exist and admissible chart in* Γ^* *such that*

(i) $\phi_i^{-1}(S \cap V_{p_i}) \subset V_{a_i}^*(\alpha)$

(ii) The admissible directions $\mathcal{A}_a$ *contains a closed subset* $\mathcal{B}_a$ *which splits* $\mathcal{A}_a$.

(iii) The star chart image $\phi^{-1}(V_p \cap S)$ *is an open star set* $V^* = V_a^*(\alpha) \cap \mathcal{B}_a$.

4.1 The tangent set

Sea M^* be a star manifold, given a point $p \in M^*$ the tangent set at p is the subset T_pM^* that can be described is the following ways:

Let $\phi : V_a^*(\alpha) \to V_p$ where $p = \phi(a)$. We write

$$T_pM^* = \phi'(a)(\mathcal{A}_\alpha(a))$$

i.e. $y \in T_pM^*$ if and only if $y = \phi'(a)h,\ h \in \mathcal{A}_\alpha(x)$.

4.2 Regular values

Let $f : D(f) \to Y$ be a mapping between $B_+ = D(f)$ and the Banach space Y. Fix a point $z \in Z$.

1. f is called a *submersion* at the point x if and only if, is a C^1 mapping (meaning that f is continuously star-differentiable) on a star-neighborhood $V_(^*x)$ of x, if $f'(x) : B \to Y$ is surjective and if the null space $N(f'(x))$ splits B. f es called a *submersion* on the set M if is a submersion at each $x \in M$.

2. The point $x \in D(f)$ is called a *regular point* of f if and only if f is a submersion at x.

3. The point $y \in Y$ is called a *regular value* if and only if f^{-1} is empty or consist of regular points. Otherwise y is called a *singular value*, i.e. $f^{-1}(y)$ contains at least one singular point.

5 The extended parametrized version of the Sard-Smale Theorem's

Consider the equation

$$H(\lambda, w) = z, \quad \lambda \in \Delta, \tag{1}$$

which depends on the parameter $w \in \Omega$

Our assumptions are:

(H1) Δ and G are manifolds, Ω is a star-manifold. This conditions are satisfied, for example, if Δ is the simplex, $G = R^{n-1}$ and $\Omega = (B_+)^n$ these are natural conditions in economic theory.

(H2) The C^k map $\Delta \times \Omega \to Z$ has z as regular point.

(H3) For each parameter $w \in \Omega_+$ the map $H(\cdot, w) : \Delta \to Z$ is a Fredholm map, where

$$ind\ H_\lambda(\lambda, w) < k.$$

for every solution $(\lambda, w) \in \Delta \times \Omega$ of equation (1)

- $H_\lambda(\lambda, w)$ is the tangent map of $H(\cdot, w) : \Delta \to z$ at the point λ.
- Weak properness. The convergence (in the star topology) $w_n \to w$ on Ω an $n \to \infty$ and $H(\lambda_n w_n) = z \ \ \forall n$ implies the existence of a convergent subsequence $\lambda_{n'} \to \lambda \in \Delta$.

Let w fixed. $H3$ implies that the solution of (1) is regular if and only if the linearization $H_\lambda(\lambda, w) : T_\lambda \Delta \to T_z Z$ is surjective.

Our main interest is to study set of solutions of the equation $e(\lambda, w) = 0$. Note for each $w \in \Omega$ the function $e(\cdot, w); \Delta \to R^{n-1}$ is a function between two manifolds of finite dimension. So $e_\lambda(\lambda, w)$ is a Fredholm operator. The existence of the derivative $e_\lambda(\lambda, w) : T_\lambda \Delta \to R^{n-1}$ generically for all $w \in \Omega_0$ is show in [**?**].

Theorem 1 (The Preimage theorem) *Let $f : M^* \to N$ a C^k mapping from a star manifold M^* and a Banach space N, if y is a regular value of f, then $S = f^{-1}(y)$ is a star-submanifold of M^**

Proof: It suffices to study the local problem. Let $V_a^*(\epsilon)$ a star neighborhood of $a \in B^+$ and consider the C^k star differentiable $\phi : B_+ \to M^*$ such that $\phi(a) = p$. Let $h \in \mathcal{A}_\epsilon(a)$ and let V_p a neighborhood of p, then $V_p \cap M^* = \phi(V_a^*(\epsilon)$, it follows that $\phi(a + \alpha h) \in V_p \cap M^*$. From the local submersion theorem if f is a submersion, there exist a parametrization ϕ such that, $\phi(a) = p, \ \phi'(a) = I$ From the definition of star manifold, for all $p' \in V_p \cap M^*$ there exist some

h and α such that $h \in \mathcal{A}_a(\alpha)$. Since *ker* $f'(p)$ splits B there exists a projection $P : B \to N$. Let $P^\perp = I - P$ and $N^\perp = P^\perp B$. Then we obtain $B = N \oplus N^\perp$ and also $f'(p) : N^\perp \to Y$ is bijective. Let its inverse be denoted by $A : Y \to N^\perp$. We define

$$a = P(p) + Af(p) \;\; and \;\; a + \alpha h = [P(p') + Af(p')]$$

From the local submersion theorem it follows that

$$f(\phi(a + \alpha h)) - f(\phi(a)) = f'(\phi(a))(\alpha h)$$

Thus the solution set of the equation $f(z) = y$ corresponds to the solution of the equation $f'(\phi(a))h = 0$. So S looks like the set $h \in \mathcal{A}_a$ such that $f'(\phi(a))h = 0$. So S is a star submanifold of M^*.

Theorem 2 (Parametrized version of the Theorem of Sard-Smale) *If (H1)- (H4) hold, then there exists an open, dense subset* Ω_0 *of* Ω *such that* z *is a regular value of* $H(\cdot, w)$*, for each parameter* $w \in W_0$

Corollary 3 *Fix an element* $w \in \Omega_0$ *If there exists a number* $n \geq 0$ *with inde* $H_x(\lambda, w)$ *for all solutions* λ *of (1), then the set of its solutions consists of an* $n-$*dimensional* C^k *Banach manifold, or the set is empty.*

Proof of the theorem: Let $M^* = \{(\lambda, w) \in \Delta \times \Omega : H(\lambda, w) = 0\}$ from the preimage theorem, (theorem 1) it follows that M^* is a star-manifold. Let $u = (\lambda, w)$ The tangent set $T_u M^*$ at the point u consists precisely of the points

$$H_\lambda(u)\lambda + H_w(u)w = 0.$$

Let $\pi : M^* \to \Omega_+$ be the nonlinear projection operator, defined by

$$\pi(\lambda, w) = w. \tag{2}$$

The operator π is G^*-differentiable with respect to u. For u fixed let $T_u M^*$ the tangent set, and define the linear operator $Q : T_u M^* \to \Omega_u$ by

$$Q(\lambda, w) = w. \tag{3}$$

The we have $Q = \pi'(\lambda, w)$. This follows because if we consider $(\lambda, w') \in T_u M^*$ then there exists a curve $t \to (\lambda, w + th)$ for some $h \in \mathcal{A}_w(\alpha)$, and $0 \leq t \leq \alpha$ on M^* If we insert this curve in equation (2), by differentiation we obtain (3).

From (H4) $\pi : M^* \to \Omega$ is proper.

On the other hand the following claim hods: Q is Fedholm operator with $ind\, Q = ind\, H_\lambda(u)$. Thus $\pi : M^* \to \Omega$ is a Fredholm operator with $ind\, \pi'_\lambda(u) = ind\, H_\lambda(u)$. Then there exists a dense subset Ω_0 of Ω such that for each $w_0 \in \Omega_0$ is a regular value of π.

Let $w_0 \in \Omega_0$, then $\pi'(u) : T_n M^* \to W$ is surjective for each $u = (\lambda_0, w_0) \in M^*$ i.e: $Q : T_u M^*$ is surjective. Since Q is surjective, if and only if $H_\lambda(u)$ is surjective, then $H_\lambda(u)\Delta \to Z$ is also surjective. Therefore 0 is a regular value of $H(\cdot, w)$.•

Proof of the claim.

(i) The operator $Q : T_u M \to \Omega$ is Fredholm with $ind\, Q = ind\, H_\lambda$.

(ii) Q is surjective if and only if H_λ is surjective.

See [**?**] lemma 78.18.