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No. 15

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Abstract

The natural projection plays a fundamental role to understand the behavior of the Walrasian economies. In this paper we extend this method to analyze the behavior of infinite dimensional economies. We introduce the definition of the social equilibrium set, and we show that there exists a bijection between this set and the Walrasian equilibria set of an infinite dimensional economy. In order to describe the main topological characteristics of both sets, we analyze the main differential characteristics of the excess utility function and then, we extend the method of the natural projection as suggested by Y. Balasko.

Keywords Natural projection, infinite dimensional economies.

JEL Classification: D50

1 Introduction

Two of the main purposes of the general equilibrium theory are to analyze the existence and the determinacy of the Walrasian equilibrium. Determinacy means local uniqueness and the finiteness of the number of distinct equilibria for a given economy. Under classical hypothesis (the canonical commodity space is the finite dimensional linear space R^n), K. Arrow and G. Debreu using Kakutani's fixed point theorem, show the generic[1] existence of the Walrasian equilibrium (see [Arrow, K.J.; Debreu, G.]), and in [Debreu, G.] the local uniqueness for almost all initial endowments is established. In the case of infinite dimensional economies the generic existence of the Walrasian equilibrium is established in [Araujo, A.; Monteiro, P.K. (93)], and its generic finiteness is provided in [Shannon, C.; Zame, B]. The issue of determinacy is relevant to comparative statics.

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[1] Roughly speaking, a generic property or a assertion is one that, except for very particular combinations of parameters, holds. More precisely: In topology and algebraic geometry, a generic property is one that holds on a dense open set, or more generally on a residual set, with the dual concept being a nowhere dense set, or more generally a meagre set.

On the other hand, in [Debreu, G.] according with the main topological characteristics of their equilibria sets, finite dimensional economies are classified in regulars and singular. In such paper, G. Debreu, shows that in the space of economies with finitely many goods, the set of regular economies is generic. In [Accinelli, E.;Puchet, M. (10)] a preliminary classification of the infinite dimensional economies is given. Most of the works in the general equilibrium refers to regular economies, by contrast, the literature on singular economies is scarce.

In this paper we show that the method of the natural projection introduced in [Balasko, Y.], can be followed, and that it is a good tool to analyze the topological properties of the equilibrium set, of finite-dimensional economies as well as for those economies, whose consumption sets are subsets of infinite dimensional vector spaces. At the same time, this approach allows us to classify infinite dimensional economies, in two groups: regulars and singular, taking account the main topological properties of their equilibrium sets. The touchstone to obtain this generalization is the method of Negishi. This approach avoids us to go into considerations on the existence of a continuous demand function (which is a not trivial fact in infinite dimensional economies, see [Araujo, A. (1987)]) and at least partially, it avoids us also, the discussion on the existence of the interior of the positive cone of consumption space. In some sense, our work can be considered as a generalization of [Chichilnisky, G.; Zhou, Y], particularly due to the fact that to establish some results, we do not requiring the assumption that the commodity space has a positive cone with non-empty interior.

Since we use the Negishi method, our approach is different from that followed by Balasko in [Balasko, Y. (1997b)] where the infinite dimensional economies (with smooth discounted utility functions), are approximated by finite-horizon truncate economies. Unlike [Shannon, C.; Zame, B] we do not assume quadratic concavity in utilities, neither any kind of substitutability between goods. However, we assume closedness condition of the utility possibility set.

With the purpose to analyze the structural behavior of the infinite dimensional economies, we introduce the concept of the social equilibria set. We show that there exists a bijection between this set and the set of the Walrasian equilibria. Once this bijection is established, we show that the natural projection can be used to analyze the topological structure of the set of Walrasian equilibria, in the same manner and with the same purposes that is pursued in the study of the classical economies. So we state that, even when the positive cone of the consumption space, has empty interior, the social equilibrium set is well defined. This property is verified by many vector spaces of significant importance for modeling the behavior of economic agents. Vector spaces such as $L^p[X,\mu]; 1 \leq p < \infty$ (i.e the set of functions with the norm $\{\int_X |f|^p d\mu\}^{\frac{1}{p}}$ is finite), verify this property. They paly a central role in many applications, where the consumption patterns have finite means and variance. In financial applications, for instance, to consider $L^2[X, \mu]$ as the consumption space is natural, and we interpret $x(t)$ as the consumption if state t occurs (see [Duffie, D.]). Much of the properties of the equilibrium set of the economies with this property are largely unexplored.

At the same time, we show that even in cases where the emptiness of the positive cone is verified, the Pareto optimal allocations do not change very much if the fundamentals of the economy do not change much. However to characterize the social equilibrium set as a Banach manifold, we need to restrict ourselves to the cases of vector spaces whose positive cones have non-empty interior. As a corollary, this approach allow us to classify (and to characterize) the infinite dimensional economies in two types: regulars and singular, and to show that economies of the first type conform a generic set also for such economies. Under these conditions, the potentiality of the method of the natural projection to analyze, in an unified way, these two types of different economies appear clearly. To make apparent this fact is perhaps the most significant contribution of this work

The paper is organized as follows: In the next section we introduce the model. This represents an infinite dimensional exchange economy. In section 3 we introduce the concept of social equilibrium set. In section 4 we consider the concept of natural projection, as a function with domain in the social equilibrium set. Next in section 5 we analyze the structure of the social equilibrium from a differential topological point of view. This is the main section of this paper. Finally we give some comments and conclusions. In all the paper we follows as close as possible the Balasko approach, given in [Balasko, Y.].

2 The Arrow-Debreu Model: An extension to Inifinite Dimensional Economies

Let n denote the (finite) number of consumers. Consumers i's are represented by endowments w_i and utility functions, u_i. The consumption set is B_+, the positive cone of a Banach Lattice B. We denote by $B_{++} = \{x \in B_+ : x \neq 0\}$. Consumer i is endowed with a commodity bundle $w_i \geq k_i > 0$, meaning that each consumer have some positive amount of endowments. The endowment vector is denoted by $w = (w_1, ..., w_n) \in \Omega_+$. We use the notation $\Omega = B^n_+$ and $\Omega_+ = B^n_{++}$. As usual we say that the element $k \in B$ is positive, and we write $k > 0$, if and only if $k \in B_+$ and $k \neq 0$. Consequently $x \geq y$ means that $x - y \in B_+$. The following definition is standard:

Definition 1 *We say that $x = (x_1, ...x_n)$ with $x_i \in B_+ \ \forall \ i \in \{1, ..., n\}$ is an admissible allocation if and only if: $\sum_{i_1}^n x_i \leq \sum_{i=1}^n w_i$. The aggregate endowment is $W = \sum_{i=1}^n w_i$.*

So, each economy is represented by the symbology: $\mathcal{E} = \{B_+, u_i, w_i, I\}$ where $I = \{1, ...n\}$ is a set of index, one for each consumer.

Since the interior of a positive cone of a Banach space, can be empty, in order to define the Gateaux-derivative of the utility functions, we introduce the concept of admissible direction. Note that under our assumptions w_i is a point in the positive cone, but no necessarily an interior point.

Definition 2 *We say that a direction h is admissible for $x \in B_+$ if $||h|| = 1$ and there exist $\alpha > 0$ such that $x + \alpha h \in B_+$. Note that if h is admissible for $x \in B_+$ then the interval*

$$[x, x + \alpha h] = \{y \in B : y = x + \beta h, 0 \leq \beta \leq \alpha\} \subset X.$$

Utilities functions $u_i : B_+ \to R$ are positive functions, i.e: $u_i(x) \in R_+$, satisfying the following assumptions:

i. Strictly monotony, i.e: if $x \geq y$ and $x \neq y$ then $u(x) > u(y)$. In particular it follows that for every $i \in I$ if $y \in B_+/B_{++}$ the $u_i(x) > u_i(y) \ \ \forall\, x \in B_{++}$.

ii. Gateaux differentiable up to any order in every admissible direction.

iii. The second Gateaux derivative of each utility function is a continuous linear operator from B to $L(X, R)$ i.e: $u''(x) \in L(B, L(B, R))$ and negative defined, i.e: $hu''(x)h < 0$, for all $h \in B$. As usual by $L(X, Y)$ we symbolize the set of linear operator from X to Y.

By $\Delta^n = \{\lambda \in R^n : \sum_{i=1}^n \lambda_i = 1,\ \lambda_i \geq 0\ \forall i\}$ we denote the simplex $(n-1)-$dimensional. Following the Negishi approach, for each $\lambda \in \Delta^n$, we introduce the particular simple social planner's utility function (or aggregate utility function) $u : B_+ \to R$ defined by:

$$u_\lambda(x) = \sum_{i=1}^{m} \lambda_i u_i(x)$$

The utility possibility set plays in this work a central role. This set is symbolized by $\mathcal{U} \subset R^n_+$

$$\mathcal{U} = \left\{ u \in R^n_+ : u_i = u_i(x_i)\ i = 1, ..., n; \text{for some allocation } x : \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} w_i \right\}.$$

We assume that the utility possibility set is closed. Recall that this property is verified for instance if the ordered interval is weakly compact, (see [Mas-Colell, A.; Zame, W.R.]). However it is possible to give examples of economies verifying closedness condition, without weakly compactness of the ordered interval. Myopic utility functions is a milder sufficient condition for closedness condition (see for instance [Araujo, A. (85)]). The following assertions holds and are well know:

- From he monotonicity of preferences it follows that $\mathcal{U}$ is bounded above by $(u_1(W), ..., u_n(W))$ where $W = \sum_{i=1}^n w_i$.

- If the utility functions are strictly concave, then the utility possibility set is a convex subset in R^n.

- If the utility possibility set is closed, and utilities are strictly monotone functions then, the following assertions hold:

  1. The set of Pareto optimal allocations is not empty.
  2. For each Pareto optimal allocation x the utility vector $u = (u_1(x_1), ..., u_n(x_n)$ is in the boundary of $\mathcal{U}$.

     For a discussion on the existence of Pareto optimal allocations, see [Accinelli, E. (02)].

The following theorem summarize this properties:

Theorem 1 *For fixed $\lambda \in \Delta^n$, if utilities are strictly monotone functions and the utility possibilities is closed, then the solution of the maximization problem*

$$\begin{array}{c} \max_{x\in\Omega} \sum_{i=1}^n \lambda_i u_i(x) \\ s.t. \sum_{i=1}^n x_i = \sum_{i=1}^n w_i \end{array} \tag{1}$$

exists, and it is a Pareto optimal allocation.

Proof: If the utility possibility is closed and bounded, for any $\lambda \in \Delta^n$, there exists u_λ in $\mathcal{U}$ such $\lambda u_\lambda \geq \lambda u, \ \forall\ u \in \mathcal{U}$. Since $u_\lambda \in \mathcal{U}$ there exist a feasible allocation x_λ such that $u_{\lambda i} = u_i(x_{\lambda i})\ i = 1, ..., n$. Given that utilities are strictly monotone functions, u_λ belong to the boundary of $\mathcal{U}$, and so, x_λ is a Pareto optimal allocation. •

Theorem 2 *If utility functions are strictly concave then for each $\lambda \in \Delta^n$ the solution of the maximization problem (1) is unique. Since, in addition, we assume strict monotonicity of the utility functions this solution is in the boundary of the utility possibility set.*

Proof: If the utility functions $u_i : B_+ \to R$ are strictly concave, then its convex combination $u : \Omega \to R$ is also a strictly concave function, and so, the maximum of $u(x)$ in the set of feasible allocation is reached at a single point. On the other hand, since utilities are strictly monotone, this allocation verify $\sum_{i=1}^m x_i = \sum_{i=1}^m w_i$.•

We symbolize by $\mathcal{PO}$ the set of feasible Pareto optimal allocations.

Theorem 3 *If the utility possibility set is closed and utilities are strictly concave and monotone then, for each feasible Pareto optimal allocation* $\bar{x}$, *there exists* $\bar{\lambda} \in \Delta^n$ *such that* $u(\bar{x}) \geq \sum_{i=1}^{n} \bar{\lambda}_i u_i(x_i) \ \ \forall\, x \in \mathcal{PO}$.

Proof: Under the hypothesis of the theorem, $\mathcal{U}$ is a convex subset of R^n. Since utilities are monotone $u(\bar{x})$ belong to the boundary of $\mathcal{U}$. From the convex separation theorem there exists $\bar{\lambda} \in \Delta^m$ such that $\bar{\lambda} u(\bar{x}) \geq \bar{\lambda} u \ \ \forall u \in \mathcal{U}$.•

Let $x(\lambda, w) \in \Omega$ be the allocation solving the maximization problem (1) by an economy which initial endowments are given by $w \in \Omega_+$.

Let us now introduce our main tool to analyze the characteristics of the equilibrium set. This is the individual excess utility function: $e_i : \Delta^m \times \Omega \to R$ defined by:

$$e_i(\lambda, w) = u_i'(x_i(\lambda, w))[x_i(\lambda, w) - w_i],$$

under the hypothesis considered this solution exists and is unique for each $(\lambda, w) \in \Delta^n \times \Omega_+$.

The excess utility function: $e : \Delta^n \times \Omega_+ \to R^n$ is defined by:

$$e(\lambda, w) = (e_1(\lambda, w), ..., e_n(\lambda, w))$$

Note that the excess utility function satisfies $\lambda e(\lambda, w) = 0$ and homogeneity of 0 degree i.e.: $e(\alpha\lambda) = e(\alpha)\ \forall \alpha > 0$.

3 The Equilibrium Social Set

One of the main issue of this work, is to introduce and to analyze the topological structure of the equilibrium social set, because as it may be show, there exists a bijection between this set and the set of Walrasian equilibrium of a given economy. As we shall see, it is easier to work with this set than with the set of Walrasian equilibria. Fortunately, many of the results obtained working with this set, can be trivially extended to the set of Walrasian equilibria (see theorem (4).

We define the equilibrium social set E is the subset of $\Delta^n \times \Omega_+$ defined by $e(\lambda, w) = 0$ thus:

$$E = \{(\lambda, w) \in \Delta^n \times \Omega_+ :\ e(\lambda, w) = 0\}.$$

Note that for fixed w and $\lambda \in \Delta^n$ verifying $e(\lambda, w) = 0$ the pair $(\lambda_j u_j'(x_j(\lambda, w)), x(\lambda)$ is a Walrasian equilibrium, and reciprocally for each Walrasian equilibrium (p, x) of the economy with utilities given by u_i and endowments w, there exists $\lambda \in \Delta^n$ such that $x(\lambda, w)$ solves the maximization problem (1) below).

Proposition 1 *If the endowment w verify the conditions $w_i > 0$ for all $i \in 1, ..., n$ and utilities are strictly monotone functions then, for all $\lambda \in \Delta^n$ such that $(\lambda, w) \in E$, the inequalities $\lambda_i > 0, \forall i = 1, ..., m,$ hold.*

Proof: Note that under the hypothesis considered if $x(\lambda, w) \in E$ and solves the maximization problem (1) then $x_i(\lambda, w) > 0$ for each coordinate of $x(\lambda, w)$ because $x_i(\lambda, w) \succeq w_i \geq k_i > 0$ and all bundle set in B_{++} is preferable to 0. Then $\lambda_i > 0$ for all i, because if $\lambda_i = 0$ for some $i \in \{1, ..., m\}$ them $x_i(\lambda, w) = 0$ and $u_i(0) < u_i(w_i)$ so, $x(\lambda, w)$ is not an allocation of equilibrium.•

Remark 1 *Since $w_i \geq k_i > 0$ for all $i \in I$ there exist some $\alpha_i > 0$ such that if $(\lambda, w) \in E$, then $\lambda_i \geq \alpha_i > 0$. Let $\alpha = min\,\{\alpha_i, i \in I\}$ be the minimum value for $\lambda_i : (\lambda, w) \in E$ and $w_i \geq k_i > 0$. Moreover, if λ_i is too small then the solution of the maximization problem (1) verify de inequality $u_i(x_i(\lambda, w) \leq u_i(w_i)$ and so, this can not be an equilibrium solution.*

The following proposition will be useful in the next section:

Proposition 2 *The embedding map $E \to \Delta^n \times \Omega_+$ is continuous.*

The assertion follows because the embedding map is the identity from $\Delta^n \times \Omega_+$ restricted to the subset E.

The next theorem asserts that the set of social equilibria E and the set $\mathcal{W}$ of Walrasian equilibria are isomorphs. This property allow us to work with the social equilibria set, to analyze the topological structure of the Walrasian equilibrium set and thus use the main tools of the differential calculus. Note that fixed w, the set of $\lambda \in E$ is a subset of R^n.

Theorem 4 *There exists a bijection between the equilibrium social set E and the set of Walrasian equilibrium $\mathcal{W}$.*

Proof: Let $(\lambda, w) \in E$, it follows that $x(\lambda, w)$ verify the condition

$$e_i(\lambda, w) = u_i'(x_i(\lambda, w))[x_i(\lambda, w) - w_i] = 0.$$

Since $x(\lambda, w)$ is a Pareto optimal allocation, from the first order conditions, for the maximization problem (1) it follows that $\lambda_i u_i'(x_i(\lambda, w)) = \lambda_j u_j'(x_j(\lambda, w))) = p(\lambda, w)$ for all $i, j \in I$, being $p(\lambda, w)$ the lagrange mutiplier for this problem. The existence of the Lagrange multiplier for such maximization program is showed in [Araujo, A. and Monteiro P.K. (1990)]. Since $u'(x) : B \to R$ is a linear functional, then it follows that $p(\lambda, w) \in B^*$ (the dual space of B). Since utilities are strictly monotone $p(\lambda, w) \neq 0$ so $(x(\lambda, w), p(\lambda, w)) \in \mathcal{W}$. Reciprocally fixed de economy w, for each $(x, p) \in \mathcal{W}$ there exist λ such that $(\lambda, w) \in E$. Since x a is Pareto optimal there exist $\lambda \in \Delta^n$ such that x solves the maximization problem (1).•

Theorem 5 *Under the hypothesis of our model, the individual excess utility function $e_i(\cdot, w) : \Delta^n \to R$ is differentiable.*

Proof For each $\lambda \in \Delta^n_+ = \{\lambda \in \Delta^m : \lambda_i > 0 \ \forall i = 1, ..., n\}$ and $W = \sum_{i=1}^m w_i \in B_{++}$, the solution of the maximization problem (1) verify the first order condition. So the following identities hold:

$$[\lambda_i u_i'(x_i) - p]h = 0, \forall \ h \text{ admissible and } i = 1, ..., n$$

$$\textstyle\sum_{i=1}^m x_i - W = 0$$

We define for each $i \in I$ $F_i : \Delta^n \times B_+ \times B^* \to B^*$ by the law $F_i(\lambda, x_i, W) = \lambda_i u_i'(x_i) - p$ and $F : \Omega \times B_{++} \to B$ by the law: $F(x, W) = \sum_{i=1}^m x_i - W$.

From the generalized implicit function theorem see [Accinelli, E. (10)], we know that there exist a relative neighborhood $V*_{\lambda,W} \subset \Delta^n \times B_+$ of $(\lambda, W) \in E$ and a G-derivable functions (implicit functions), $x : V^*_{\lambda,W} \to B^n$ and $p : V^*_{\lambda,W} \to B^*$ verifying:

$$F_i(\lambda, x_i(\lambda, W), W)h = [\lambda_i u_i'(x_i(\lambda, W)) - p(\lambda, W)]h = 0, \ i = 1, ..., m,$$

$$F(x(\lambda, W)), W) = \textstyle\sum_{i=1}^m x_i(\lambda, W)) - W = 0.\bullet$$

As a conclusion of this theorem it follows that Pareto optimal allocations change continuously with λ and W.

With the aim of simplify the reading of the following sections we recall some mathematical definitions.

Recall that if M and N are manifolds, and $f : M \to N$:

1. f is called a *submersion* at x if and only if $f'(x)$ is onto, and the null space $N(f'(x))$ splits TM_x.

2. A point $x \in M$ is a *regular point of* f is and only if f is a submersion at x. Otherwise x is called a singular point.

3. A point $z \in N$ is a *regular value of* f if and only if f^{-1} is empty or consists only of regular points.

Now we introduce the definitions of Fredholm operator and Fredholm map:

Definition 3 *Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator. T is said to be Fredholm if the following hold.*

1. *$Ker(T)$ is finite dimensional. Where by $Ker(T)$ we symbolize the subset $\{x \in X : Tx = 0\}$*
2. *$Ran(T)$ is closed. By $Ran(T)$ we symbolize the subset $\{y \in Y : \exists x \in X, t(x) = y\}$.*
3. *$Coker(T)$ is finite dimensional. Recall that the $Coker(T)$ is the quotient space $Y/Ran(T)$.*

Let M and N two Banach Manifolds. A map $f : M \to N$ is a Fredholm map *if $f'_x : TM_x \to TN_y$ is a linear Fredholm operator. Where by TM_x and TM_y we symbolize the tangent spaces of M and N at x and y respectively, being $y = f(x)$.*

The next proposition shows that the Equilibrium Social Set, is (generically) a Banach manifold. In order to establish this result we restrict ourselves to consider consumption spaces whose positive cones have non-empty interior.

Remark 2 *It is important to say that to assume that the consumption space is a positive cone with non-empty interior, is certainly a restriction. Spaces like L^p, $1 \leq p < \infty$ has not this property. Nevertheless many economies can be modeled using topological spaces whose positive cone has non-empty interior. Typically, examples of this kind of spaces are the following: The space of functions k times differentiable $C^k(X, R)$; L_∞ i.e; the set of essentially bounded real functions and the set l_∞ of bounded sequences, all of them are widely used in economic theory. For instance L_∞ and l_∞ play an important role in growth theory and in general in intertemporal allocation problems, which are themselves at the heart of a rich body of economic theory (see [Prescot, E.C.; Mera, R.]). The natural commodity bundles for these problems are consumption streams. If we consider consumption of a single physical commodity taking place continuously trough time we are led to consider L_∞. On the other hand, if the consumption takes place at discrete intervals over an infinite horizon, we are led naturally to consider the space l_∞. We interpret each element $c \in l_\infty$ as a discrete consumption stream, and $c(t)$ the consumption in the $t-th$ period. If in addition, we consider that the consumption streams verify some differential properties,like existence of derivatives or differentiability, then the space $C^k(X, R)$, is capable of application for a wide variety of specifics models.*

Notation: By e_{Ω_0} we symbolize the restriction of e to $int[\Delta^n] \times \Omega_0$ being $\Omega_0 \subset \Omega_+$

Proposition 3 *If the positive cone of the consumption space, has a non-empty interior, then there exist a dense and open subset $\Omega_0 \subset \Omega_+$ such that the set of solutions of the equation $\bar{e}_{\Omega_0}(\lambda, w) = 0$ is a Banach manifold in $\Delta^n \times \Omega_0$.*

This Banach manifold is symbolized by

$$E_{\Omega_0} = \{(\lambda, w) \in \Delta^n \times \Omega_0 : e(\lambda, w) = 0\}$$

and represents the social equilibria manifold.

Given that the proof of the proposition involve several and well differentiated arguments, we will give the prove according with a sequence of several steps and necessary considerations.

1. Since $\lambda e(\lambda, w) = 0$ without lost of generality we can consider that $\lambda_n \neq 0$ and then $e_n(\lambda, w) = -\frac{\lambda_1}{\lambda_n} e_1(\lambda, w) - .... - \frac{\lambda_{n-1}}{\lambda_n} e_{n-1}(\lambda, w)$. Then we define the equivalent map $\bar{e} : \Delta^n \times \Omega_+ \to R^{n-1}$, such that the first n coordinates of $e(\lambda, w)$ are equal to the $n-1$ coordinates of $\bar{e}(\lambda, w)$.

2. The interior of the simplex $(n-1)-$dimensional, symbolized by $int[\Delta^n]$, is a nonempty manifold of dimension $n-1$ in R^n.

3. The set Ω_+ is an open and nonempty subset of the Banach space B^n.

4. The product $int[\Delta^n] \times \Omega_0$ for each open subset $\Omega_0 \subseteq \Omega_+$ is a Banach manifold in $R^n \times B^n$.

5. There exist an open and dense subset Ω_0 of Ω_+ such that the map $\bar{e} : int[\Delta^n] \times \Omega_0 \to R^{n-1}$ is a submersion. This assertion is the result of the following facts:

   (a) $\bar{e}'(\lambda, w) : T(int[\Delta] \times \Omega_+)_{(\lambda,w)} \to R^{n-1}$ is onto and the null space $N(\bar{e}'(\lambda, w)$ splits $T(int[\Delta^n] \times \Omega_+)_{(\lambda,w)}$[2], see appendix.

   (b) For each $w \in \Omega_+$ the mapping $\bar{e}(\cdot, w) : int[\Delta^n] \to R^{n-1}$ is Fredholm of index zero.

   For a detailed prove see appendix.

6. Now using theorem 73.c in [Zeidler, E. (1993)] section 73-11 it follows that: $E_{\Omega_0} = e^{-1}_{\Omega_0}(0)$ is a Banach manifold.

   Or equivalently the set $E_{\Omega_0} = \{(\lambda, w) \in int[\Delta^n] \times \Omega_0 : e(\lambda, w) = 0\}$ is a Banach manifold.

In the next section using the natural projection approach as in [Balasko, Y.] we will show that there exists an open and dense subset $\Omega_{00} \subset \Omega_0$ of Ω_+ such that 0 is a regular value of $\bar{e}(\cdot, w)$ for each parameter $w \in \Omega_{00}$. Moreover, we will show that for each parameter $w \in \Omega_{00}$ the equation $\bar{e}(\cdot, w) = 0$, has at most finitely many solutions.

This means that the set of regular economies $w \in \Omega$ is generic and that generically, i.e: for each economy $w \in \Omega_{00}$, the set of λ such that $(\lambda, w) \in E$ is finite.

[2]By $T(int[\Delta^n] \times \Omega_+)_{(\lambda,w)}$ we symbolize the tangent to the manifold $int[\Delta^n] \times \Omega_0$ at the point (λ, w). Note that $\bar{e}'(\lambda, w)(u, v)) = \frac{\partial \bar{e}}{\partial \lambda} u + \frac{\partial \bar{e}}{\partial w} v$.

4 The Natural Projection

The natural projection is the map $\pi : E \to \Omega_+$ such that for each $w \in \Omega_+$ $\pi^{-1}(w) = (\lambda, w) \in E$. So, if we consider utilities as given and characterize the economy by the endowments w, the preimage of the natural projection is the equilibrium set of the economy $w \in \Omega_+$. The fact that the subset E_{Ω_0} possesses the structure of a Banach manifold, enable us to use the notion of smooth mapping for the natural projection $\pi : E_{\Omega_0} \to \Omega$. Thus the natural projection is a good tool to understand the behavior of the equilibrium manifold for infinite dimensional economies, as well as for the finite dimensional economies.

Proposition 4 *The natural projection $\pi : E_{\Omega_0} \to \Omega$ is continuous.*

Proof: The natural projection is the composition of two maps: The embedding $E \to \Delta^n \times \Omega$ which is continuous, and the projection $\Delta^n \times \Omega_+ \to \Omega$ which is smooth because the coordinate functions of this map are smooth.•

Recall that a function $f : X \to Y$ between two topological spaces is proper if and only if the preimage of every compact set in Y is compact in X.

Proposition 5 *The natural projection $\pi : E_{\Omega_0} \to \Omega$ is a proper map.*

Proof: Let H be a compact subset of Ω, let $\{(\lambda^k, w^k)\}_{k \in N}$ be a sequence in $\pi^{-1}(H) \subset E$. Thus, $(\lambda^k, w^k) \in \pi^{-1}(w^k)$. Since H is compact there exists a subsequence $w^{k_i} \to w \in H$. Let $\{(\lambda^{k_i}, w^{k_i})\}$ be a sequence in E, such that $(\lambda^{k_i}, w^{k_i}) \in \pi^{-1}(w^{k_i})$. Since $[\alpha, 1-\alpha]^n$ is a compact subset in R^n there exists a subsequence $\lambda^{k_{ij}} \to \lambda$ and $1 - \alpha \geq \lambda_i \geq \alpha > 0$, see remark (1).

Let $\{(\lambda^{k_{ij}}, w^{k_{ij}})\}$ be the corresponding subsequence in E_{Ω_0}. Thus for each k_{ij} it follows that $e(\lambda^{k_{ij}}, w^{k_{ij}}) = 0$. Since $\bar{e} : \Delta^n, \times \Omega$ is a continuous function then $\bar{e}(\lambda, w) = 0$ so, $(\lambda, w) \in E$. We conclude that for each sequence in $\pi^{-1}(H)$ there exist a convergent subsequence, so $\pi^{-1}(H)$ is a compact subset in E_{Ω_0}.•

Taking an starting point the equilibria social set, the combination of smoothness and properness of the natural projection is sufficient to extend the characterization of regular economies made in [Balasko, Y.] to economies with infinitely many commodities and a finite number of consumers.

Now we have all the mathematical tool in our hand to deal with one of the main issue of this paper. In the next section we consider that that the interior of the positive cone of the consumption space in non-empty

5 Regular and singular economies

Let assume that the Banach space B, has the property of the non-empty interior of the positive cone. In this case, from theorem (3), there exists a residual set $\Omega_0 \subset \Omega$ such that the social equilibrium set E is a Banach Manifold immerse in $\Delta^n \times \Omega_0$.

Let $E(\mathcal{R})$ denote the property for the equilibrium $(\lambda, w) \in E$ to be a regular point of the projection $\pi : E \to \Omega$. The elements of $E(\mathcal{R})$ are known as regular equilibria. We denote by $E(\Gamma)$ the set of critical equilibria in E. This two sets $E(\mathcal{R})$ and $E(\Gamma)$ are complementary subsets in E i.e,: $E = E(\mathcal{R}) \cup E(\Sigma)$ and $E(\mathcal{R}) \cap E(\Sigma) = \emptyset$. We will say that an economy $w \in \Omega_0$ is regular if and only if w is a regular value of $\pi : E \to \Omega$ and singular in other case. The set of regular economies will be symbolized by $\mathcal{R}$ and its complements by Σ the set of singular economies.

Now we will characterize the regular points of the projection $\pi : E_{\Omega_0} \to \Omega$. We will proceed by steps

1. For a fixed x we let TE_x the tangent space at the point $x = (\lambda, w) \in E_{\Omega_0}$.
2. We define $Q : TE_x \to \Omega$ through $Q(\lambda, w) = w$.
3. Then we have $Q(x) = \pi'(x)\ \forall x \in E$ where $\pi'_x(\lambda, w) = \pi'_\lambda \lambda + \pi'_w w$ being $\pi'_\lambda \equiv 0$ and $\pi'_w \equiv I$.
4. We know that the operator π is proper.
5. The operator $Q : TE_x \to \Omega$ is Fredholm. Because
   (a) The dimension of the kernel of Q verify $dimKer(Q) = dimKer(\pi_x) = n - 1$. More precisely consists of all points $(v, z) \in R^{n-1} \times B^n$ with $\pi(v, z) = 0$, i.e, $\pi_\lambda v + \pi_w z = 0$ recall that $\pi'_\lambda = 0$ and $\pi'_w = I$.
   (b) $codimR(Q) = codimR(\pi'(x)) = 0$
6. It follows that the nonlinear operator $\pi : E_{\Omega_0} \to \Omega$ is Fredholm.
7. Then, from the parameterized version of the Theorem of Sard-Smale, (see [Zeidler, E. (1993)] vol 4, section 78.10) it follows that there exists a an open and dense subset $\Omega_{00} \subset \Omega_0$ of Ω_+ such that w is a regular value of $\pi(\cdot, w)$, for each parameter $w \in \Omega_{00}$.

Theorem 6 *For every $w \in \Omega_{00}$, the set $\pi^{-1}(w)$ is finite.*

Proof: Let $w \in \Omega_{00} \subset \Omega_+$ and, $(\lambda, w) \in \pi^{-1}(w)$. So $(\lambda, w) \in E$, if and only if $\lambda \in \Delta^{n-1}$ verify the equation $\bar{e}(\lambda, w) = 0$. Since, for each $w \in W_{00}$, 0 is a regular value of $\bar{e}$ this equation has a finite set of solutions, then the proposition follows.•

Fixing utilities, economies are characterized by endowments. So Ω_+ can be considered as the space of economies. We say that an economy is regular if w is a regular value for the projection $\pi(\lambda, w)$. From the Sard-Smale theorem, it is possible to conclude that the set of regular economies $\mathcal{R}$, is a dense and open subset of Ω. and consequently, the complementary set Σ, of economies is a meager subset of Ω, an this constitute the set of singular economies. So, the economy w is regular if $\pi^{-1}(w) \subseteq E(\mathcal{R})$ and singular if there exist some $(\lambda, w) \in \pi^{-1}(w) : (\lambda, w) \in E(\Sigma)$. The next relationships hold: $\Omega = \mathcal{R} \cup \Sigma$ and $\mathcal{R} \cap \Sigma = \emptyset$.

6 Conclusions and further research

The concurrent use of the utility function "a la Negishi" (see [Negishi, T.]) and the natural projection "a la Balasko" (see [Balasko, Y.]), allow us to extend the analysis of the structure of the set of Walrasian equilibria of economies "a la Arrow-Debreu", to the analysis of economies with infinitely many goods. Moreover the conjunction of these two approaches enables us to consider in an unified way both types of economies.

The introduction of the social equilibrium set plays a central role to make such extension possible. This set can be naturally considered in the case of finite economies, so the study of both types of economies can be achieved by an unique method. However the social equilibrium set plays an important role to understand de topological structure of the Walrasian equilibrium set of the infinite dimensional economies.

Finally, let us say that if large changes in the fundamentals of a regular economy does not happen, then big changes in the behavior of the economy can not be expected. However, if a small change in the fundamentals of a singular economy occurs, then big changes in the behavior of the economy can be expected. We think that in this point focusses the importance of a set as small as the singular economies. Most general equilibrium works, focuses on the study of the behavior of regular economies and so far the behavior of singular economies are not well known. We believe that, despite its smallness, this set plays an important role in economics, particularly when it comes to explaining unexpected and abrupt changes in economic performance.

Appendix

In this section we prove assertion (v) of the theorem (3). Basically this prove is given in [Accinelli, E.;Puchet, M. (10)] to make it more accessible to the reader is reproduced here.

Let $w \in \Omega_+$ be the initial endowments of the economy and let $h = (h_1, ...h_n)$ be a vector in B^n such that h_i is admissible. Consider the vector $v = (v_1, ..., v_n) \in R^n$ such that $w_i + v_i h_i \in \Omega_+$

and $v_n h_n = \sum_{i_1}^{n-1} v_i h_i$ and define $\eta = (\eta_1, \eta_2, ..., \eta_n)$ verifying that $\eta_i = v_i h_i; i = 1, 2, ..., n-1$. The vector η will be thought as a parameters for redistributions of initial endowments.

The excess utility function for the economy $\mathcal{E}(\eta) = \{u_i, w(\eta)_i, I\}$ will be:

$$e(\lambda, w(\eta)) = (e_1(\lambda, w_1 + v_1 h_1), ..., e_n(\lambda, w_n + v_n h_n)) \,, \tag{2}$$

where

$$e_i(\lambda, w_i + v_i h_i) = u_i'(x_i^*(\lambda, W))[x_i(\lambda, W) - w_i - v_i h_i].$$

Observe that the function $e_i(\lambda, w_i + v_i h_i)$ $i = 1, 2..., n$ depends only on the $n-1$ real variables $v_i, i = 1, ..., n-1$. So we can consider the equivalent excess utility function $\bar{e}(v_1, ..., v_{n-1}) = \bar{e}(v)$, observe that $\bar{e} : R^{(n-1)} \to R^{n-1}$.

The derivative of $\bar{e}_i$ with respect to $v_i, i = 1, ..., n-1$ evaluated at $(\lambda, w(\eta))$ is given by:

$$\frac{\partial e_i(\lambda, w_i + v_i h)}{\partial v_i} = \frac{\partial \tilde{e}_i(v_i)}{\partial v_i} = -u_i'(x_i(\lambda, W)) h_i,$$

$$\frac{\partial e_n(\lambda, w_n - \sum_{i=1}^{n-1} v_i h_i)}{\partial v_i} = u_n'(x_n(\lambda, W)) h_i.$$

Then it follows that:

$$\frac{\partial e(\lambda, w(\eta))}{\partial v_i} = \frac{\partial \tilde{e}(v)}{\partial v_i} = (0, ..., 0, \frac{\partial \tilde{e}_i(v_i)}{\partial v_i} h_i, 0..., 0, \frac{\partial \tilde{e_n}(w_n - \sum_{i=1}^{n-1} v_i h_i)}{\partial v_i} h_i)$$

$$= (0, ..., 0, -u_i'(x_i(\lambda, W)) h_i, 0, ..., 0, u_n'(x_n(\lambda, W)) h_i).$$

Let $\bar{e} : R^{n-1} \to R^{n-1}$ be the function defined by the $n-1$ first coordinates of $\tilde{e}$, i.e:

$$\bar{e}(\lambda, w + \eta) = (e_1(\lambda, v_1 h_1), ..., e_{n-1}(\lambda, v_{n-1} h_{n-1}) = (\tilde{e_1}(v_1), ..., \tilde{e}_{n-1}(v_{n-1})).$$

Then:

$$\frac{\partial \bar{e}}{\partial \bar{v}}(\lambda, w(v)) = -\begin{bmatrix} u_1'(x_1^*) & 0 & \dots & 0 \\ 0 & u_2'(x_2^*) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & u_{n-1}'(x_{n-1}^*) \end{bmatrix} \in L\left(X^{(n-1)}, R^{(n-1)}\right). \tag{3}$$

The rank of this matrix is equal to $n-1$, as the rank of a matrix is locally invariant, then for all w there exists an arbitrarily close vector $w(\eta)$: such that the rank of $\frac{\partial \bar{e}}{\partial \bar{v}}(\lambda, w(v))$ is equal to $n-1$ this prove the denseness of Ω_0.[3] Let $\Delta_w = \{\lambda \in int[\Delta] : u_i(x(\lambda)) \geq u_i(w_i)\}$ be the set of the *individually rational social weights*. Then for a given $\epsilon > 0$ there exists $\delta > 0$ such that if

[3] This shows that the linearized form of $e(\lambda, w)$ with respect to λ, $e_\lambda(\lambda, w) : T\Delta_\lambda \to R^{n-1}$ is surjective. Then we can use the surjective implicit function theorem.

$|e_i(\lambda, w(\eta) - e_i(\lambda, w)| \leq \|u_i'\| \|h_i\| < \epsilon$ for $h_i : \|h_i\| < \delta$, where $\|u_i'\| = sup_{h_1 : \|h_i\| = 1} |u_i'(x(\lambda, W)|, \lambda \in \Delta_w$, i.e. the excess utility function of the perturbed economy is in a neighborhood of the excess utility function of the original one.

To show that zero is a regular value for e we need to prove that $Ker(e')$ splits $R^{n-1} \times \Omega$. In our case, as the image of the function e is a subset of R^{n-1}, (i.e.; $e(int[\Delta] \times \Omega_+) \subseteq R^{n-1}$) so the quotient space $(R^{n-1} \times \Omega)/Ker(e')$ has finite dimension, then $codim[Ker(e')] < \infty$ and the splitting property is automatically satisfied[4] [Zeidler, E. (1993)].

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[4] Recall that in a locally convex Hausdorff space X, every finite dimensional subspace Y can be complemented, that means that there exists a closed vector subspace Z such that $X = Y \oplus Z$ i.e. Y splits X see [**?**]

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